ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
FLORIAN LUCA

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PROBLEMS PROPOSED IN THIS ISSUE

H-673 Proposed by H.-J. Seiffert, Berlin, Germany

The Pell and Pell-Lucas numbers are defined by

$$P_0 = 0, \quad P_1 = 1, \text{ and } P_{n+1} = 2P_n + P_{n-1} \text{ for } n \geq 1,$$
$$Q_0 = 2, \quad Q_1 = 2, \text{ and } Q_{n+1} = 2Q_n + Q_{n-1} \text{ for } n \geq 1,$$

respectively. Prove that, for all positive integers n,

$$P_{2n-1} = 2^{-n} \sum_{k=0}^{2n-1} (-1)^{\lfloor (2n-5k-5)/4 \rfloor} \binom{4n-1}{k},$$
$$Q_{2n} = 2^{1-n} \sum_{k=0}^{2n} (-1)^{\lfloor (2n-5k)/4 \rfloor} \binom{4n+1}{k}.$$

H-674 Proposed by José Luis Díaz-Barrero, Barcelona, Spain and Pantelimon George Popescu, Bucharest, Romania

Let n be a positive integer. Prove that

$$n\pi^2 F_n F_{n+1} \leq (n(F_n - 1) + \pi(F_{n+2} - 1))^2.$$

H-675 Proposed by John J. Jaroma, Ave Maria, Florida

An odd perfect number is an odd integer that is equal to the sum of its proper divisors. Although such a number is currently unknown, many conditions necessary for its existence have been established. The earliest is attributed to Euler who showed that if n is an odd perfect number then

$$n = p^\alpha p_1^{2\beta_1} \cdots p_r^{2\beta_r},$$

where $p, p_1, \ldots, p_r$ are distinct odd primes and $p \equiv \alpha \equiv 1 \pmod 4$. The prime p has been dubbed the *special prime*. Show that the least prime divisor of n is not p.

H-676 Proposed by Mohammad K. Azarian, Evansville, Indiana

Let $f(x) = \sinh x$, $g(x) = \ln(x + \sqrt{1 + x^2})$, and $h(x) = 1/(2 - f(-g(-x)))$. Also, let $h_0(x) = h(x)$, $h_1(x) = h_0(h_0(x)), \ldots$, and $h_{n+1}(x) = h_0(h_n(x))$ for all $n \geq 0$. If $p(x) = \prod_{i=0}^{n} h_i(x)$, then find the coefficient of F_k^r $(k > 0)$ in the expansion of $1/\sqrt{p(F_k)}$ in terms of r and n.

SOLUTIONS

Inequalities With Weighted Power Sums

H-654 Proposed by Slavko Simic, Belgrade, Yugoslavia (Vol. 45, No. 2, May 2007)

Let $x = \{x_i\}_{i=1}^{n}$ be a sequence of real numbers and $p = \{p_i\}_{i=1}^{n}$ be a sequence of positive numbers with $\sum_{i=1}^{n} p_i = 1$. Define $S_k = \sum_{i=1}^{k} p_i x_i^k - \left(\sum_{i=1}^{k} p_i x_i\right)^k$, for $k = 1, 2, 3, \ldots$. Prove that $S_3^2 \leq \frac{3}{2} S_2 S_4$. Is it true that the inequality $S_{2m+1}^{2m} \leq \frac{(2m+1)m^{2m}}{(m+1)^{2m-1}} S_2 S_{2m+2}^{2m-1}$ holds for all $m \geq 1$?

Partial solution by the proposer

We give a simple proof of the first inequality. Namely, it is well-known that $S_4 \geq 0$ for arbitrary x and p because the function $x \mapsto x^4$ is convex. Making a shift $x \longmapsto x + t$ with an arbitrary real number t, we have

$$S_4(t) := \sum_{i=1}^{4} p_i (x_i + t)^4 - \left(\sum_{i=1}^{4} p_i (x_i + t)\right)^4 = \sum_{i=1}^{4} p_i (x_i + t)^4 - \left(\sum_{i=1}^{4} p_i x_i + t\right)^4.$$

Furthermore, $S_4(t) \geq 0$ for all real numbers t. Developing in powers of t, we get

$$S_4(t) = S_4 + 4S_3 t + 6S_2 t^2.$$

Putting $t := -S_3/3S_2$ and using the fact that $S_4(t) \geq 0$ for this value of t, we obtain the assertion from the part 1.

No solution was received for the inequality suggested at part 2 although Paul S. Bruckman showed, using Hölder's inequality, that the stronger inequality

$$S_{2m+1}^{2m} \leq S_2 S_{2m+2}^{2m-1}$$

holds for all $m = 1, 2, \ldots$ and for sequences x and p such that $\sum_{i=1}^{n} p_i x_i = 0$.

More Inequalities With Weighted Power Sums

H-655 Proposed by Slavko Simic, Belgrade, Yugoslavia (Vol. 45, No. 2, May 2007)

Let $\{c_i\}_{i=1}^{n}$ be a finite sequence of distinct positive integers and $q > 1$ be a natural number. Prove that $\left\lfloor \frac{\sum_{i=1}^{n} c_i q^{c_i}}{\sum_{i=1}^{n} q^{c_i}} \right\rfloor = c$, where $c = \max\{c_i : i = 1, \ldots, n\}$. Is it true that $\left\lfloor \frac{(q-1)\sum_{i=1}^{n} c_i q^{c_i}}{\sum_{i=1}^{n} q^{c_i}} \right\rfloor = c(q-1) - 1$?

Partial solution by the proposer

We shall give a simple proof of the first part of the problem valid for all real $q \geq 2$. Since $n > 1$, we have

$$\frac{\sum_{i=1}^n c_i q^{c_i}}{\sum_{i=1}^n q^{c_i}} < \max\{c_i : i = 1, \dots, n\} \frac{\sum_{i=1}^n q^{c_i}}{\sum_{i=1}^n q^{c_i}} = c.$$

Also since

$$\sum_{n=1}^{\infty} \frac{n-1}{q^n} = \frac{1}{(q-1)^2},$$

we get that

$$\sum_{i:c_i<c} \frac{c-c_i-1}{q^{c-c_i}} < \frac{1}{(q-1)^2};$$

i.e.,

$$(c-1)\sum_{i:c_i<c} q^{c_i} - \sum_{i:c_i<c} c_i q^{c_i} < \frac{q^c}{(q-1)^2}.$$

Hence,

$$(c-1)\sum_{i:c_i\leq c} q^{c_i} - \sum_{i:c_i\leq c} c_i q^{c_i} < \frac{q^c}{(q-1)^2} + (c-1)q^c - cq^c = q^c\left(\frac{1}{(q-1)^2} - 1\right) \leq 0.$$

Therefore,

$$c - 1 < \frac{\sum_{i=1}^n c_i q^{c_i}}{\sum_{i=1}^n q^{c_i}} < c.$$

Observe that on the right hand side we need to exclude the case $n = 1$ for which the strict inequality becomes equality. The conclusion of part 1 now follows.

No solution was received for the inequality proposed in part 2. The proposer claims that it follows in an analogous way as the proof of part 1 but the argument needs a closer examination.

Also solved partially by Paul S. Bruckman.

A Sequence Tending To e

H-656 Proposed by Andrew Cusumano, Great Neck, NY (Vol. 45, No. 2, May 2007)

Let $A_n = \sum_{k=1}^n k^k$. Show that $\lim_{n\to\infty}\left(\frac{A_{n+2}}{A_{n+1}} - \frac{A_{n+1}}{A_n}\right) = e$. Show that the same holds for the sequence of general term $A_n = (n+1)^{n+1} - n^n$.

Solution by the editor based on a solution by G.-C. Greubel, Newport News, VA

We start with the first part. Let Φ_n be given by

$$\Phi_n := \frac{A_{n+2}}{A_{n+1}} - \frac{A_{n+1}}{A_n}.$$

It is clear that $A_n = n^n + A_{n-1}$. Now the ratio of A_{n+2}/A_{n+1} is given by

$$\begin{aligned}\frac{A_{n+2}}{A_{n+1}} &= \frac{(n+2)^{(n+2)} + A_{n+1}}{A_{n+1}} = \frac{(n+2)^{(n+2)}}{(n+1)^{(n+1)} + A_n} + 1 \\ &= (n+2)\left(\frac{n+2}{n+1}\right)^{n+1}\left(1 + \frac{A_n}{(n+1)^{(n+1)}}\right)^{-1} + 1 \\ &= (n+2)\left(1 + \frac{1}{n+1}\right)^{n+1}\left(1 + \frac{A_n}{(n+1)^{(n+1)}}\right)^{-1} + 1.\end{aligned}$$

It is easy to see that $A_n = n^n(1 + o(1))$ as $n \to \infty$. Thus,

$$\frac{A_n}{(n+1)^{n+1}} = \frac{n^n(1+o(1))}{(n+1)^{n+1}} = \frac{1+o(1)}{e(n+1)} \qquad \text{as} \quad n \to \infty.$$

With these estimates, Φ_n becomes

$$\begin{aligned}\Phi_n &= (n+2)\left(1 + \frac{1}{n+1}\right)^{n+1}\left(1 + \frac{1}{e(n+1)} + o\left(\frac{1}{n}\right)\right)^{-1} \\ &\quad -(n+1)\left(1 + \frac{1}{n}\right)^{n}\left(1 + \frac{1}{en} + o\left(\frac{1}{n}\right)\right)^{-1} \qquad \text{as} \quad n \to \infty.\end{aligned}$$

By taking the limit as $n \to \infty$ of both sides above we are lead to

$$\begin{aligned}\lim_{n\to\infty} \Phi_n &= \lim_{n\to\infty}\left\{(n+2)\left(1 + \frac{1}{n+1}\right)^{n+1}\left(1 + \frac{1}{en} + o\left(\frac{1}{n}\right)\right)^{-1}\right. \\ &\quad \left. -(n+1)\left(1 + \frac{1}{n}\right)^{n}\left(1 + \frac{1}{en} + o\left(\frac{1}{n}\right)\right)^{-1}\right\} \\ &= \lim_{n\to\infty}\left\{\frac{(n+2)^{n+2}}{(n+1)^{n+1}} - \frac{(n+1)^{n+1}}{n^n} + o(1)\right\} \\ &= e,\end{aligned}$$

where the last limit above is due to Brothers and Knox [1]. This is the desired result for part 1 of the problem.

We now deal with part 2. Write

$$\begin{aligned}\Psi_n &:= \frac{A_{n+2}}{A_{n+1}} = \frac{(n+3)^{(n+3)} - (n+2)^{(n+2)}}{(n+2)^{(n+2)} - (n+1)^{(n+1)}} \\ &= (n+3)\left(1 + \frac{1}{n+1}\right)^{n+1}\frac{\left(1+\frac{1}{n+2}\right)^{n+2} - \frac{1}{n+3}}{\left(1+\frac{1}{n+1}\right)^{n+1} - \frac{1}{n+2}}.\end{aligned}$$

Using the known asymptotic

$$\left(1 + \frac{1}{n}\right)^n = e - \frac{e}{2n} + o\left(\frac{1}{n}\right) \qquad \text{as} \quad n \to \infty,$$

(see (4) in [1]), it follows easily that

$$\frac{\left(1+\frac{1}{n+2}\right)^{n+2} - \frac{1}{n+3}}{\left(1+\frac{1}{n+1}\right)^{n+1} - \frac{1}{n+2}} = \frac{1 - \frac{1}{2(n+2)} - \frac{1}{e(n+3)} + o\left(\frac{1}{n}\right)}{1 - \frac{1}{2(n+1)} - \frac{1}{e(n+2)} + o\left(\frac{1}{n}\right)} = 1 + o\left(\frac{1}{n}\right) \qquad \text{as} \quad n \to \infty.$$

Thus, the limiting value of Ψ_n is

$$\begin{aligned}\lim_{n\to\infty} \Psi_n &= \lim_{n\to\infty} \left\{ (n+3)\left(1+\frac{1}{n+1}\right)^{n+1}\left(1+o\left(\frac{1}{n}\right)\right)\right. \\ &- \left.(n+2)\left(1+\frac{1}{n}\right)^{n}\left(1+o\left(\frac{1}{n}\right)\right)\right\} \\ &= \lim_{n\to\infty} \left\{ \left(\frac{(n+2)^{n+2}}{(n+1)^{n+1}} - \frac{(n+1)^{n+1}}{n^n}\right)\right. \\ &+ \left.\left(\left(1+\frac{1}{n+1}\right)^{n+1} - \left(1+\frac{1}{n}\right)^{n}\right) + o(1)\right\} \\ &= e,\end{aligned}$$

where the above limit follows again from the result of [1]. This is the desired result for part 2 of the problem.

Also solved by Paul S. Bruckman.

[1] H. J. Brothers and J. A. Knox, *New Closed-Form Approximations to the Logarithmic Constant e*, Math. Intell., **20** (1998), 25–29.

Fermat's Last Theorem and the Golden Section

H-657 Proposed by Paul S. Bruckman, Sointula, Canada (Vol. 45, No. 2, May 2007)

Show that the equation $(a+b\alpha)^4 + (a+b\beta)^4 = c^4$ has no nonzero integer solutions a, b, c, where $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$.

Solution by the proposer

By expansion, the given Diophantine equation can be put in the form

$$2a^4 + 4a^3b + 18a^2b^2 + 16ab^3 + 7b^4 = c^4.$$

Since the equation is homogeneous, we may suppose that the $\gcd(a,b,c) = 1$. Multiplying the above equation by 3 and regrouping we get

$$5a^4 + (a+3b)^4 = 3c^4 + 60ab^3 + 60b^4.$$

Reducing the above equation modulo 4 we get

$$a^4 + (a-b)^4 + c^4 \equiv 0 \pmod 4.$$

We see that this is possible only if all three a, b and c are even, which is a contradiction.

Also solved by G. C. Greubel.

The Cauchy-Schwarz Inequality and Fibonacci Numbers

H-658 Proposed by José Luis Díaz-Barrero, Barcelona, Spain (Vol. 45, No. 3, August 2007)

Let n be a positive integer. Prove that

$$F_{2n} < \frac{1}{2}\Big(\frac{2^n F_n F_{n+1}}{F_{n+2}-1} + \binom{2n}{n}\frac{F_{n+2}-1}{2^n}\Big).$$

Solution by H.-J. Seiffert, Berlin, Germany

In view of the arithmetic-geometric inequality, it suffices to show that

$$F_{2n} < \sqrt{\binom{2n}{n} F_n F_{n+1}} \qquad \text{for} \quad n > 1.$$

In (1) and (2) of [1], it is shown that

$$F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_k.$$

The charming identity

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

is a particular case of the well-known Vandermonde convolution formula. According to equation (I_3) in [2], it holds that

$$F_n F_{n+1} = \sum_{k=0}^{n} F_k^2.$$

Therefore the desired inequality follows immediately from the Cauchy-Schwarz inequality. Equality is excluded because the corresponding vectors are linearly independent, as is easily seen (for example, the first component of the vector with Fibonacci entries is 0 while the first component of the vector with binomial coefficient entries is 1) .

Also solved by Paul S. Bruckman, Kenneth B. Davenport and the proposer.

[1] P. Haukkanen, *On a Binomial Sum for the Fibonacci and Related Numbers*, The Fibonacci Quarterly, **34.4** (1996), 326–331.

[2] V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*, Santa Clara, CA, The Fibonacci Association, 1979.

Errata. In H-669, the identity to be proved should have been

$$\sum_{n=0}^{\infty}\left[\frac{1}{5n+1} + \frac{2}{5n+2} + \frac{\beta^2}{5n+3} + \frac{\beta}{5n+4} - \frac{\beta^2}{5n+5}\right](-1)^n \beta^{5n} = \pi\left(\frac{\alpha^2}{5}\right)^{\frac{3}{4}}.$$

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