ADVANCED PROBLEMS AND SOLUTIONS
EDITED BY
FLORIAN LUCA

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, IMATE, UNAM, AP. POSTAL 61-3 (XANGARI), CP 58 089, MORE-LIA, MICHOACAN, MEXICO, or by e-mail at flucamatmor.unam.mx as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-673 Proposed by H.-J. Seiffert, Berlin, Germany
The Pell and Pell-Lucas numbers are defined by

\[ P_0 = 0, \quad P_1 = 1, \quad \text{and} \quad P_{n+1} = 2P_n + P_{n-1} \quad \text{for} \quad n \geq 1, \]
\[ Q_0 = 2, \quad Q_1 = 2, \quad \text{and} \quad Q_{n+1} = 2Q_n + Q_{n-1} \quad \text{for} \quad n \geq 1, \]
respectively. Prove that, for all positive integers \( n \),

\[ P_{2n-1} = 2^{-n} \sum_{k=0}^{2n-1} (-1)^{(2n-5k-5)/4} \binom{4n - 1}{k}, \]
\[ Q_{2n} = 2^{1-n} \sum_{k=0}^{2n} (-1)^{(2n-5k)/4} \binom{4n + 1}{k}. \]

H-674 Proposed by José Luis Díaz-Barrero, Barcelona, Spain and Pantelimon George Popescu, Bucharest, Romania
Let \( n \) be a positive integer. Prove that

\[ n \pi^2 F_n F_{n+1} \leq (n(F_n - 1) + \pi(F_{n+2} - 1))^2. \]

H-675 Proposed by John J. Jaroma, Ave Maria, Florida
An odd perfect number is an odd integer that is equal to the sum of its proper divisors. Although such a number is currently unknown, many conditions necessary for its existence have been established. The earliest is attributed to Euler who showed that if \( n \) is an odd perfect number then

\[ n = p^\alpha p_1^{2\beta_1} \cdots p_r^{2\beta_r}, \]
where \( p, p_1, \ldots, p_r \) are distinct odd primes and \( p \equiv \alpha \equiv 1 \pmod{4} \). The prime \( p \) has been dubbed the \textit{special prime}. Show that the least prime divisor of \( n \) is not \( p \).
THE FIBONACCI QUARTERLY

H-676 Proposed by Mohammad K. Azarian, Evansville, Indiana

Let \( f(x) = \sinh x \), \( g(x) = \ln(x + \sqrt{1+x^2}) \), and \( h(x) = 1/(2 - f(-g(-x))) \). Also, let \( h_0(x) = b(x) \), \( h_1(x) = h_0(h_0(x)) \), \ldots, and \( h_{n+1}(x) = h_0(h_n(x)) \) for all \( n \geq 0 \). If \( p(x) = \prod_{i=0}^{n} h_i(x) \), then find the coefficient of \( F_k^r \) (k > 0) in the expansion of \( 1/\sqrt{p(F_k)} \) in terms of \( r \) and \( n \).

SOLUTIONS

Inequalities With Weighted Power Sums

H-654 Proposed by Slavko Simic, Belgrade, Yugoslavia

(Vol. 45, No. 2, May 2007)

Let \( x = \{x_i\}_{i=1}^{n} \) be a sequence of real numbers and \( p = \{p_i\}_{i=1}^{n} \) be a sequence of positive numbers with \( \sum_{i=1}^{n} p_i = 1 \). Define \( S_k = \sum_{i=1}^{k} p_ix_i^k - (\sum_{i=1}^{k} p_ix_i) \), for \( k = 1, 2, 3, \ldots \). Prove that \( S_3^2 \leq \frac{3}{2} S_2 S_4 \). Is it true that the inequality \( S_{2m+1}^2 \leq \frac{2m+1}{m+1} S_2 S_{2m+2} \) holds for all \( m \geq 1 \)?

Partial solution by the proposer

We give a simple proof of the first inequality. Namely, it is well-known that \( S_1 \geq 0 \) for arbitrary \( x \) and \( p \) because the function \( x \mapsto x^4 \) is convex. Making a shift \( x \mapsto x + t \) with an arbitrary real number \( t \), we have

\[
S_4(t) := \sum_{i=1}^{4} p_i(x_i + t)^4 - \left(\sum_{i=1}^{4} p_i(x_i + t)\right)^4 = \sum_{i=1}^{4} p_i(x_i + t)^4 - \left(\sum_{i=1}^{4} p_ix_i + t\right)^4.
\]

Furthermore, \( S_4(t) \geq 0 \) for all real numbers \( t \). Developing in powers of \( t \), we get

\[
S_4(t) = S_4 + 4S_3t + 6S_2t^2.
\]

Putting \( t := -S_3/3S_2 \) and using the fact that \( S_4(t) \geq 0 \) for this value of \( t \), we obtain the assertion from the part 1.

No solution was received for the inequality suggested at part 2 although Paul S. Bruckman showed, using Hölder’s inequality, that the stronger inequality

\[
S_{2m+1}^2 \leq S_2 S_{2m+2}
\]

holds for all \( m = 1, 2, \ldots \) and for sequences \( x \) and \( p \) such that \( \sum_{i=1}^{n} p_ix_i = 0 \).

More Inequalities With Weighted Power Sums

H-655 Proposed by Slavko Simic, Belgrade, Yugoslavia

(Vol. 45, No. 2, May 2007)

Let \( \{c_i\}_{i=1}^{n} \) be a finite sequence of distinct positive integers and \( q > 1 \) be a natural number. Prove that \( \left[ \frac{\sum_{i=1}^{n} c_i^q}{\sum_{i=1}^{n} c_i} \right] = c \), where \( c = \max\{c_i : i = 1, \ldots, n\} \). Is it true that \( \frac{(q-1)\sum_{i=1}^{n} c_i^q}{\sum_{i=1}^{n} c_i} = c(q-1) - 1 \)?

284 VOLUME 46/47, NUMBER 3
Partial solution by the proposer

We shall give a simple proof of the first part of the problem valid for all real $q \geq 2$. Since $n > 1$, we have

$$\frac{\sum_{i=1}^{n} c_i q^{c_i}}{\sum_{i=1}^{n} q^{c_i}} < \max\{c_i : i = 1, \ldots, n\} \frac{\sum_{i=1}^{n} q^{c_i}}{\sum_{i=1}^{n} q^{c_i}} = c.$$  

Also since

$$\sum_{n=1}^{\infty} \frac{n-1}{q^n} = \frac{1}{(q-1)^2},$$

we get that

$$\sum_{i \in \mathbb{N}_< c} \frac{c - c_i - 1}{q^{c-c_i}} < \frac{1}{(q-1)^2};$$

i.e.,

$$(c-1) \sum_{i \in \mathbb{N}_< c} q^{c_i} - \sum_{i \in \mathbb{N}_< c} c_i q^{c_i} < \frac{q^c}{(q-1)^2}.$$  

Hence,

$$\left( (c-1) \sum_{i \in \mathbb{N}_< c} q^{c_i} - \sum_{i \in \mathbb{N}_< c} c_i q^{c_i} \right) - \frac{q^c}{(q-1)^2} = q^c \left( \frac{1}{(q-1)^2} - 1 \right) \leq 0.$$  

Therefore,

$$c - 1 < \sum_{i=1}^{n} \frac{c_i q^{c_i}}{\sum_{i=1}^{n} q^{c_i}} < c.$$  

Observe that on the right hand side we need to exclude the case $n = 1$ for which the strict inequality becomes equality. The conclusion of part 1 now follows.

No solution was received for the inequality proposed in part 2. The proposer claims that it follows in an analogous way as the proof of part 1 but the argument needs a closer examination.

Also solved partially by Paul S. Bruckman.

A Sequence Tending To $e$

H-656 Proposed by Andrew Cusumano, Great Neck, NY  
(Vol. 45, No. 2, May 2007)

Let $A_n = \sum_{k=1}^{n} k^h$. Show that $\lim_{n \to \infty} \left( \frac{A_{n+2}}{A_{n+1}} - \frac{A_{n+1}}{A_n} \right) = e$. Show that the same holds for the sequence of general term $A_n = (n+1)^{n+1} - n^n$.

Solution by the editor based on a solution by G.-C. Greubel, Newport News, VA

We start with the first part. Let $\Phi_n$ be given by

$$\Phi_n := \frac{A_{n+2}}{A_{n+1}} - \frac{A_{n+1}}{A_n}.$$  

AUGUST 2008/2009 285
It is clear that \( A_n = n^n + A_{n-1} \). Now the ratio of \( A_{n+2}/A_{n+1} \) is given by

\[
\frac{A_{n+2}}{A_{n+1}} = \frac{(n+2)^{(n+2)}}{(n+2)^{(n+1)}} + \frac{A_{n+1}}{A_{n}} + 1
\]

\[
= (n+2) \left( \frac{n+2}{n+1} \right)^{n+1} \left( 1 + \frac{A_n}{(n+1)^{(n+1)}} \right)^{-1} + 1
\]

\[
= (n+2) \left( \frac{1 + \frac{1}{n+1}}{n+1} \right)^{n+1} \left( 1 + \frac{A_n}{(n+1)^{(n+1)}} \right)^{-1} + 1.
\]

It is easy to see that \( A_n = n^n(1 + o(1)) \) as \( n \to \infty \). Thus,

\[
\frac{A_n}{(n+1)^{n+1}} = \frac{n^n(1 + o(1))}{e(n+1)} = 1 + o(1)
\]

as \( n \to \infty \).

With these estimates, \( \Phi_n \) becomes

\[
\Phi_n = (n+2) \left( 1 + \frac{1}{n+1} \right)^{n+1} \left( 1 + \frac{1}{e(n+1) + o \left( \frac{1}{n} \right)} \right)^{-1}
\]

\[
- (n+1) \left( 1 + \frac{1}{n} \right)^{n+1} \left( 1 + \frac{1}{en + o \left( \frac{1}{n} \right)} \right)^{-1}
\]

\[
\lim_{n \to \infty} \Phi_n = \lim_{n \to \infty} \left\{ (n+2) \left( 1 + \frac{1}{n+1} \right)^{n+1} \left( 1 + \frac{1}{en + o \left( \frac{1}{n} \right)} \right)^{-1}
\]

\[
- (n+1) \left( 1 + \frac{1}{n} \right)^{n+1} \left( 1 + \frac{1}{en + o \left( \frac{1}{n} \right)} \right)^{-1}\right\}
\]

\[
= \lim_{n \to \infty} \left\{ (n+2)^{(n+2)} - (n+1)^{(n+1)} \right\}
\]

\[
= e,
\]

where the last limit above is due to Brothers and Knox [1]. This is the desired result for part 1 of the problem.

We now deal with part 2. Write

\[
\Psi_n := \frac{A_{n+2}}{A_{n+1}} = \frac{(n+3)^{(n+3)} - (n+2)^{(n+2)}}{(n+2)^{(n+2)} - (n+1)^{(n+1)}}
\]

\[
= (n+3) \left( 1 + \frac{1}{n+1} \right)^{n+1} \left( 1 + \frac{1}{en + o \left( \frac{1}{n} \right)} \right)^{-1} - \frac{1}{n+3}.
\]

Using the known asymptotic

\[
\left( 1 + \frac{1}{n} \right)^{n+1} = e - \frac{e}{2n} + o \left( \frac{1}{n} \right)
\]

as \( n \to \infty \), (see (4) in [1]), it follows easily that

\[
\frac{1 + \frac{1}{n+1}}{(1 + \frac{1}{n+1})^{n+1} - \frac{1}{n+2}^{n+1}} = 1 + \frac{1}{2(n+2)} - \frac{1}{e(n+3)} + o \left( \frac{1}{n} \right)
\]

\[
= 1 + o \left( \frac{1}{n} \right)
\]

as \( n \to \infty \).
Thus, the limiting value of $\Psi_n$ is

$$
\lim_{n \to \infty} \Psi_n = \lim_{n \to \infty} \left\{ (n + 3) \left( 1 + \frac{1}{n + 1} \right)^{n+1} \left( 1 + o \left( \frac{1}{n} \right) \right) - (n + 2) \left( 1 + \frac{1}{n} \right)^n \left( 1 + o \left( \frac{1}{n} \right) \right) \right\}
$$

$$
= \lim_{n \to \infty} \left\{ \frac{(n + 2)^{n+2}}{(n + 1)^{n+1}} \frac{(n + 1)^{n+1}}{n^n} + \left( 1 + \frac{1}{n + 1} \right)^{n+1} - \left( 1 + \frac{1}{n} \right)^n + o(1) \right\}
$$

$$
= e,
$$

where the above limit follows again from the result of [1]. This is the desired result for part 2 of the problem.

Also solved by Paul S. Bruckman.


**Fermat’s Last Theorem and the Golden Section**

**H-657** Proposed by Paul S. Bruckman, Sointula, Canada (Vol. 45, No. 2, May 2007)

Show that the equation $(a + b\alpha)^4 + (a + b\beta)^4 = c^4$ has no nonzero integer solutions $a, b, c$, where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$.

**Solution by the proposer**

By expansion, the given Diophantine equation can be put in the form

$$
2a^4 + 4a^3b + 18a^2b^2 + 16ab^3 + 7b^4 = c^4.
$$

Since the equation is homogeneous, we may suppose that the $\gcd(a, b, c) = 1$. Multiplying the above equation by 3 and regrouping we get

$$
5a^4 + (a + 3b)^4 = 3c^4 + 60ab^3 + 60b^4.
$$

Reducing the above equation modulo 4 we get

$$
a^4 + (a - b)^4 + c^4 \equiv 0 \pmod{4}.
$$

We see that this is possible only if all three $a$, $b$ and $c$ are even, which is a contradiction.

Also solved by G. C. Greubel.
Let \( n \) be a positive integer. Prove that
\[
F_{2n} < \frac{1}{2} \left( \frac{2^n F_n F_{n+1}}{F_{n+2}} - 1 \right) + \binom{2n}{n} \left( \frac{F_{n+2} - 1}{2^n} \right).
\]

Solution by H.-J. Seiffert, Berlin, Germany

In view of the arithmetic-geometric inequality, it suffices to show that
\[
F_{2n} < \sqrt{\binom{2n}{n} F_n F_{n+1}} \quad \text{for} \quad n > 1.
\]

In (1) and (2) of [1], it is shown that
\[
F_{2n} = \sum_{k=0}^{n} \binom{n}{k} F_k.
\]
The charming identity
\[
\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2
\]
is a particular case of the well-known Vandermonde convolution formula. According to equation \((I_3)\) in [2], it holds that
\[
F_n F_{n+1} = \sum_{k=0}^{n} F_k^2.
\]
Therefore the desired inequality follows immediately from the Cauchy-Schwarz inequality. Equality is excluded because the corresponding vectors are linearly independent, as is easily seen (for example, the first component of the vector with Fibonacci entries is 0 while the first component of the vector with binomial coefficient entries is 1).

Also solved by Paul S. Bruckman, Kenneth B. Davenport and the proposer.


Errata. In H-669, the identity to be proved should have been
\[
\sum_{n=0}^{\infty} \left[ \frac{1}{5n + 1} + \frac{2}{5n + 2} + \frac{\beta^2}{5n + 3} + \frac{\beta}{5n + 4} - \frac{\beta^2}{5n + 5} \right] (-1)^n \beta^n = \pi \left( \alpha^2 \right)^{\frac{3}{2}}.
\]

PLEASE SEND IN PROPOSALS!