ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by February 15, 2007. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \; F_0 = 0, \; F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n, \; L_0 = 2, \; L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2, \; \beta = (1-\sqrt{5})/2, \; F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1015 (Correction) **Proposed by José Luis Díaz-Barrero and Miquel Grau-Sánchez, Universidad Politécnica de Cataluña, Barcelona, Spain**

Let n be a positive integer. Prove that

$$\left(\sum_{k=1}^{n} F_k F_{2k}\right) \left(\sum_{k=1}^{n} \frac{F_k^2}{\sqrt{L_k}}\right)^2 \geq F_n^3 F_{n+1}^3.$$

B-1017 (Correction) **Proposed by M.N. Deshpande, Nagpur, India**

Define $\{a_n\}$ by $a_1 = a_2 = 0, \; a_3 = a_4 = 1$ and

$$a_n = a_{n-1} + a_{n-3} + a_{n-4} + k(n)$$

for $n \geq 5$ where

$$k(n) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ i^{n-2} & \text{if } n \text{ is even} \end{cases}$$

and $i = \sqrt{-1}$.

Prove or disprove: $a_n + 2a_{n+2} + a_{n+4}$ is a Fibonacci number for all integers $n \geq 1$.

B-1019 Proposed by Hiroshi Matsui, Naoki Saita, Kazuki Kawata, Yusuke Sakurama, and Ryohei Miyadera, Kwansei Gakuin University, Nishinomiya, Japan

(a) Define $\{a_n\}$ by $a_1 = a_2 = 1$ and

$$a_n = a_{n-1} + a_{n-2} + \begin{cases} 1 & \text{if } n \equiv 1 \pmod 4 \\ 0 & \text{if } n \not\equiv 1 \pmod 4 \end{cases}$$

for $n \geq 3$. Express a_n in terms of F_n.

(b) Prove that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \alpha$.

B-1020 Proposed by H.-J. Seiffert, Berlin, Germany

(a) Let $(A_j)_{j\geq 0}$ be any sequence of numbers such that $A_j \neq 0$ and $A_{j+1} = A_j + A_{j-1}$ for $j \geq 1$. Prove that, for all positive integers n,

$$A_1 \sum_{k=1}^{n} \left[A_0^{n-k} F_k \prod_{j=k}^{n} \frac{F_j}{A_j} \right] = F_n.$$

(b) Deduce the identities

$$\sum_{k=1}^{n} \left[2^{n-k} F_k \prod_{j=k}^{n} \frac{F_j}{L_j} \right] = F_n$$

and

$$3 \sum_{k=1}^{n} 2^{n-k} F_k^2 F_{k+1} F_{k+2} = F_n F_{n+1} F_{n+2} F_{n+3}.$$

SOLUTIONS

A Tough Sum

B-1005 Proposed by H.-J. Seiffert, Berlin, Germany (Vol. 43, no. 3, August 2005)

Prove that, for all integers k and n with $0 \leq k \leq n$,

$$\sum_{j=0}^{2n-2k} (-1)^j \binom{2n+1}{j} \binom{2n-k-j}{k} F_j = 0.$$

Solution by Paul S. Bruckman, Sointula, BC, Canada

Let $A(n,k)$ denote the given expression, and let $B(x,n) = \sum_{k=0}^{n} A(n,k)x^k$. Therefore

$$B(x,n) = \sum_{j=0}^{2n}(-1)^j F_j \binom{2n+1}{j} \sum_{0 \le 2k \le 2n-j} \binom{2n-k-j}{k} x^k.$$

The inner sum is equal to $\phi_{2n-j+1}(x)$, where

$$\phi_m(x) = (r^m - s^m)/(r-s), \; m = 0, \; 1, \; 2, \cdots, \tag{1}$$

with

$$r = r(x) = (1+\theta)/2, \; s = s(x) = (1-\theta)/2, \text{ and } \theta = \theta(x) = r - s = (1+4x)^{1/2}. \tag{2}$$

Therefore, since $\phi_0(x) = 0$, we see that

$$B(x,n) = \sum_{j=0}^{2n+1} \binom{2n+1}{j} (-1)^j F_j \quad \phi_{2n+1-j}(x). \tag{3}$$

Now $F_j \quad \phi_{2n+1-j}(x) = 1/(5^{1/2}\theta)\{\alpha^j - \beta^j\}\{r^{2n+1-j} - s^{2n+1-j}\}$; thus,

$$B(x,n) = 1/(5^{1/2}\theta) \sum_{j=0}^{2n+1} \binom{2n+1}{j}$$

$$\{r^{2n+1-j}(-\alpha)^j - r^{2n+1-j}(-\beta)^j - s^{2n+1-j}(-\alpha)^j + s^{2n+1-j}(-\beta)^j\}, \text{ or :}$$

$$B(x,n) = 1/(5^{1/2}\theta)\{(r-\alpha)^{2n+1} - (r-\beta)^{2n+1} - (s-\alpha)^{2n+1} + (s-\beta)^{2n+1}\}. \tag{4}$$

From (2), we observe that $r + s = \alpha + \beta = 1$; hence:

$$r - \alpha = -(s - \beta), \; r - \beta = -(s - \alpha). \tag{5}$$

Therefore, $B(x,n) = 1/(5^{1/2}\theta)\{-(s-\beta)^{2n+1} + (s-\beta)^{2n+1} + (s-\alpha)^{2n+1} - (s-\alpha)^{2n+1}\} = 0$. This is true for all $n \ge 0$, but also for all x. Therefore, it follows from the definition of $B(x,n)$ that $A(n,k) = 0$ for all n, k with $0 \le k \le n$.

Also solved by G.C. Greubel and the proposer.

A Sequence of Pythagorean Triangles

B-1006 Proposed by Paul S. Bruckman, Canada
(Vol. 43, no. 4, Nov. 2005)

For $n \geq 1$, let $\{A_n\}$ and $\{B_n\}$ be two sequences of positive integers denoting the lengths of the legs of a Pythagorean triangle such that $B_n = 2A_n - 2(-1)^n$. Determine $\{A_n\}$ and $\{B_n\}$ and obtain recurrence relations for these sequences.

Solution by H.-J. Seiffert, Thorwaldsenstr. 13, Berlin, Germany

For the nonnegative integer n, let $A_n = F_{n+1}F_{n+4}$ and $B_n = 2F_{n+2}F_{n+3}$. Since $F_{n+1} = F_{n+3} - F_{n+2}$ and $F_{n+4} = F_{n+3} + F_{n+2}$, we have

$$A_n^2 + B_n^2 = \left(F_{n+3}^2 - F_{n+2}^2\right)^2 + 4F_{n+2}^2F_{n+3}^2 = \left(F_{n+3}^2 + F_{n+2}^2\right)^2,$$

showing that A_n and B_n are the lengths of the legs of a Pythagorean triangle. From eqn. (3.32) of [1], it follows that $B_n = 2A_n - 2(-1)^n$. Eqns. (3.23) and (3.25) of [1] give

$$A_n = \frac{1}{5}\left(L_{2n+5} + 4(-1)^n\right) \quad \text{and} \quad B_N = \frac{2}{5}\left(L_{2n+5} - (-1)^n\right).$$

By eqn. (3.29) of [1], there holds the recurrence $L_{2n+9} = 3L_{2n+7} - L_{2n+5},\ n \geq 0$. Thus, the sequence $\{A_n\}$ satisfies the recurrence relation $A_{n+2} = 3A_{n+1} - A_n + 4(-1)^n,\ n \geq 0$, with initial values $A_0 = 3$ and $A_1 = 5$. Similarly, $\{B_n\}$ has the recurrence $B_{n+2} = 3B_{n+1} - B_n - 2(-1)^n,\ n \geq 0$, with initial values $B_0 = 4$ and $B_1 = 12$.

Remark: The sequences $\{A_n\}$ and $\{B_n\}$ are not uniquely determined by the conditions of the proposal, because one may take $A_n = F_{k(n)}F_{k(n)+3}$ and $B_n = 2F_{k(n)+1}F_{k(n)+2}$, where $\{k(n)\}$ is any sequence of positive integers such that $k(n) \equiv n + 1 \pmod 2$ for all n.

Reference:

1. A.F. Horadam & Bro. J.M. Mahon. "Pell and Pell-Lucas Polynomials." *The Fibonacci Quarterly* **23.1** (1985): 7-20.

Also solved by Brian Beasley, George C. Greubel, Russell Hendel, James Sellers, and the proposer.

Evaluate the Infinite Sum

B-1007 Proposed by Andrew Cusumano, Great Neck, New York (Vol. 43, no. 4, Nov. 2005)

Prove or disprove:

$$\frac{1}{1} + \frac{\left[(1^2+1)^N - (1^2)^N\right]}{(1\cdot 1)^N} - \frac{\left[(2^2)^N - (2^2-1)^N\right]}{(1\cdot 2)^N} + \frac{\left[(3^2+1)^N - (3^2)^N\right]}{(2\cdot 3)^N}$$

$$- \frac{\left[(5^2)^N - (5^2-1)^N\right]}{(3\cdot 5)^N} + \frac{\left[(8^2+1)^N - (8^2)^N\right]}{(5\cdot 8)^N} - \frac{\left[(13^2)^N - (13^2-1)^N\right]}{(8\cdot 13)^N} + \cdots = \alpha^N.$$

Solution by Brian D. Beasley, Presbyterian College, Clinton, SC

Using the identity $F_k^2 + (-1)^k = F_{k-1}F_{k+1}$ (see the Addendum), we express the given series as

$$1+\sum_{k=2}^{\infty} a_k := 1+\sum_{k=2}^{\infty}\left[\left(\frac{F_k^2+(-1)^k}{F_{k-1}F_k}\right)^N - \left(\frac{F_k^2}{F_{k-1}F_k}\right)^N\right] = 1+\sum_{k=2}^{\infty}\left[\left(\frac{F_{k+1}}{F_k}\right)^N - \left(\frac{F_k}{F_{k-1}}\right)^N\right].$$

This series telescopes with partial sum

$$1+\sum_{k=2}^{n}\left[\left(\frac{F_{k+1}}{F_k}\right)^N - \left(\frac{F_k}{F_{k-1}}\right)^N\right] = 1+\left[\left(\frac{F_{n+1}}{F_n}\right)^N - 1\right] = \left(\frac{F_{n+1}}{F_n}\right)^N.$$

Hence

$$1+\sum_{k=2}^{\infty} a_k = \lim_{n\to\infty}\left(\frac{F_{n+1}}{F_n}\right)^N = \alpha^N.$$

Addendum. We note that

$$F_k^2 + (-1)^k = \frac{\alpha^{2k} - 2(-1)^k + \beta^{2k}}{5} + (-1)^k = \frac{\alpha^{2k} + 3(-1)^k + \beta^{2k}}{5}$$

and

$$F_{k-1}F_{k+1} = \frac{\alpha^{2k} - (\alpha^2+\beta^2)(-1)^{k-1} + \beta^{2k}}{5} = \frac{\alpha^{2k} + 3(-1)^k + \beta^{2k}}{5}.$$

Also solved by Paul S. Bruckman, George C. Greubel, Russell J. Hendel, H.-J. Seiffert, and the proposer.

An Odd Type System

B-1008 Proposed by the Problem Editor (Vol. 43, no. 4, Nov. 2005)

Find all (a, b, c, d) that satisfy the system

$$\begin{aligned} a+b+c+d &= 0\\ ab+ac+ad+bc+bd+cd &= -3\\ abc+abd+acd+bcd &= 0\\ abcd &= 1. \end{aligned}$$

Solution by Charles K. Cook, University of South Carolina Sumter, Sumter, SC

Expanding $(x-a)(x-b)(x-c)(x-d)=0$ and using the given data, yields the equation $x^4-3x^2+1=0$.

Using the quadradic formula yields $x^2=\frac{3\pm\sqrt{5}}{2}$. Thus $x^2=\alpha^2$ and β^2.

Hence the solution is $(a,b,c,d)=(\alpha,-\alpha,\beta,-\beta)$ or any permutation thereof.

Emphasized: Kenneth Davenport noted that a similar problem appeared as problem 867 in the Fall 1995 issue of the Pi Mu Epsilon Journal.

Also solved by Brian D. Beasley, Paul S. Bruckman, George C. Greubel, Ralph Grimaldi, Roger Haskell, Russell J. Hendel, Gerald A. Heuer, Rebecca A. Hillman, H.- J. Seiffert, and the proposer.

A Fibonacci Inequality

B-1009 Proposed by José Luis Díaz-Barrero, Universitat Politècnia, de Catalunya, Barcelona, Spain (Vol. 43, no. 4, Nov. 2005)

Let n be a positive integer. Prove that

$$4+2\sum_{k=1}^{n}\left\{\frac{F_{k+1}}{\log\left(1+\frac{F_{k+1}}{F_k}\right)}\right\}<F_{n+1}+3F_{n+2}.$$

Solution by Paul S. Bruckman, P.O. Box 150, Sointula, BC V0N 3E0 (Canada)

We begin noting that $1+F_{k+1}/F_k=F_{k+2}/F_k=F_{k+2}/(F_{k+2}-F_{k+1})=(1-F_{k+1}/F_{k+2})^{-1}$. Therefore, if $S(n)$ denotes the expression in the left member of the putative inequality, we see that $S(n)=4-2\sum_{k=1}^{n}F_{k+2}(F_{k+1}/F_{k+2})/\log(1-F_{k+1}/F_{k+2})$. Let $F_{k+1}/F_{k+2}=y=y_k$. Clearly, $0<y<1$. Note that (using series expansion for instance) $\frac{-y}{\log(1-y)}<1-\frac{y}{2}$.

Therefore, $S(n)<4+2\sum_{k=1}^{n}F_{k+2}(1-F_{k+1}/2F_{k+2})=4+\sum_{k=1}^{n}(2F_{k+2}-F_{k+1})=4+\sum_{k=1}^{n}L_{k+1}=L_3+\sum_{k=1}^{n}(L_{k+3}-L_{k+2})=L_3+L_{n+3}-L_3=L_{n+3}$. On the other hand, $F_{n+1}+3F_{n+2}=F_{n+3}+2F_{n+2}=F_{n+4}+F_{n+2}=L_{n+3}$. Therefore, the desired result is proven namely: $S(n)<L_{n+3}$, for all $n\geq 1$.