ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, SCHOOL OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG 3, WITS 2050, JOHANNESBURG, SOUTH AFRICA or by e-mail at the address florian.luca@wits.ac.za as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-842 Proposed by Hideyuki Ohtsuka, Saitama, Japan

Given an integer $n \geq 0$, find a closed form expression for the sum

$$\sum_{\substack{a+b+c=n \\ a,b,c \geq 0}} F_{a+b}F_{b+c}F_{c+a}.$$

H-843 Proposed by Hideyuki Ohtsuka, Saitama, Japan

If integers a and b have the same parity with $a > b > 0$ and c is odd, show that

$$(F_a - F_b) \mid (F_{ac} - F_{bc}) \qquad \text{and} \qquad (L_a - L_b) \mid (L_{ac} - L_{bc}).$$

H-844 Proposed by Robert Frontczak, Stuttgart, Germany

Let $B_n = B_n(\alpha, \beta)$ be a generalized balancing number given by $B_0(\alpha, \beta) = \alpha$, $B_1(\alpha, \beta) = \beta$ and for $n \geq 2$,

$$B_n(\alpha, \beta) = 6B_{n-1}(\alpha, \beta) - B_{n-2}(\alpha, \beta).$$

Prove that

$$\sum_{k=0}^{2n} \binom{4n}{2k} B_{2k}(\alpha, \beta) = (2^{6n-1} + 2^{4n-1})B_{2n}(\alpha, \beta)$$

and

$$\sum_{k=0}^{\lfloor (4n-1)/2 \rfloor} \binom{4n}{2k+1} B_{2k}(\alpha, \beta) = (2^{6n-1} - 2^{4n-1})B_{2n}(\alpha, \beta).$$

H-845 **Proposed by D. M. Bătineţu, Bucharest, Romania, and N. Stanciu, Buzău, Romania**

Compute

$$\lim_{n\to\infty}\left(\lim_{x\to\infty}\left((f(x+1))^{L_n/((x+1)F_{n+1})}-(f(x))^{L_n/(xL_{n+1})}\right)x^{L_{n-1}/L_{n+1}}\right),$$

where $f:\mathbb{R}_+^*\to\mathbb{R}_+^*$ is a function that satisfies $\lim_{x\to\infty} f(x+1)/(xf(x))=a\in\mathbb{R}_+^*$.

SOLUTIONS

Evaluating the sum of a series of reciprocals

H-810 **Proposed by Ángel Plaza, Gran Canaria, Spain**
(Vol. 55, No. 3, August 2017)

Prove that

$$\sum_{n=3}^{\infty}\frac{1}{L_n^4-25}=\frac{5}{63}-\frac{1}{6\sqrt{5}}.$$

Solution by the proposer

First note that $L_n^4-25=L_{n-2}L_{n-1}L_{n+1}L_{n+2}$ and also that $L_n=\dfrac{L_{n+2}+L_{n-2}}{3}$. Therefore,

$$\frac{1}{L_n^4-25}=\frac{1/3}{L_{n-2}L_{n-1}L_nL_{n+1}}+\frac{1/3}{L_{n-1}L_nL_{n+1}L_{n+2}}. \tag{1}$$

Taking into account relation (23) in [1]:

$$\sum_{i=1}^{n-1}\frac{1}{L_iL_{i+1}L_{i+2}L_{i+3}}=-\frac{1}{8}+\frac{1}{10}\left(\frac{F_{n-1}}{L_n}+\frac{3F_n}{L_{n+1}}+\frac{F_{n+1}}{L_{n+2}}\right)$$

and letting n approach infinity, we get

$$\sum_{n=1}^{\infty}\frac{1}{L_nL_{n+1}L_{n+2}L_{n+3}}=\frac{1}{40}\left(5-2\sqrt{5}\right).$$

Therefore,

$$\sum_{n=3}^{\infty}\frac{1/3}{L_{n-2}L_{n-1}L_nL_{n+1}} = \frac{1}{120}\left(5-2\sqrt{5}\right),$$

from where the desired sum follows via (1). □

[1] R. S. Melham, *Finite sums that involve reciprocal of products of generalized Fibonacci numbers*, Integers, **13** (2013), A40.

Also solved by Brian Bradie, Dmitry Fleischman, Hideyuki Ohtsuka, and Raphael Schumacher.

Evaluating the sum of another series of reciprocals

H-811 Proposed by Ángel Plaza, Gran Canaria, Spain (Vol. 55, No. 3, August 2017)

For any positive integer k, let $\{F_{k,n}\}_{n\geq 0}$ be defined by $F_{k,n+2} = kF_{k,n+1} + F_{k,n}$ for $n \geq 0$ with $F_{k,0} = 0$, $F_{k,1} = 1$. Prove that

$$\sum_{n=0}^{\infty} \frac{1}{1+F_{k,2n+1}} = \frac{\sqrt{k^2+4}}{2k}.$$

Solution by Brian Bradie

Let k be a positive integer, and put $\alpha_k := (k+\sqrt{k^2+4})/2$. Then,

$$F_{k,n} = \frac{1}{\sqrt{k^2+4}}\left(\alpha_k^n - \left(-\frac{1}{\alpha_k}\right)^n\right) \quad \text{and} \quad F_{k,2n+1} = \frac{1}{\sqrt{k^2+4}}\left(\alpha_k^{2n+1} + \frac{1}{\alpha_k^{2n+1}}\right).$$

Moreover,

$$\begin{aligned} \frac{1}{1+F_{k,2n+1}} &= \frac{\sqrt{k^2+4}\,\alpha_k^{2n+1}}{\alpha_k^{4n+2} + \sqrt{k^2+4}\,\alpha_k^{2n+1} + 1} \\ &= \frac{\sqrt{k^2+4}}{k}\left(\frac{\alpha_k}{\alpha_k^{2n+1}+\alpha_k} - \frac{\frac{1}{\alpha_k}}{\alpha_k^{2n+1}+\frac{1}{\alpha_k}}\right). \end{aligned}$$

Since

$$\frac{\alpha_k}{\alpha_k^{2n+3}+\alpha_k} = \frac{\frac{1}{\alpha_k}}{\alpha_k^{2n+1}+\frac{1}{\alpha_k}},$$

it follows that the desired series telescopes and so its value is

$$\sum_{n=0}^{\infty} \frac{1}{1+F_{k,2n+1}} = \frac{\sqrt{k^2+4}}{k} \cdot \frac{\alpha_k}{\alpha_k+\alpha_k} = \frac{\sqrt{k^2+4}}{2k}.$$

Also solved by Dmitry Fleischman, Hideyuki Ohtsuka, Raphael Schumacher, and the proposer.

An identity with sums of products of binomial coefficients

H-812 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 55, No. 3, August 2017)

Prove that

$$\sum_{i+j=F_{n+1}} \binom{F_{n+1}}{i}\binom{F_n}{i}\binom{L_n}{j} = \sum_{i+j=F_{n+1}} \binom{F_n}{i}\binom{L_n}{i}\binom{2j}{F_{n+1}}.$$

Solution by the proposer

Let $a = F_{n+1}$, $b = F_n$, and $c = L_n$. Note that $b + c = 2a$. We have

$$\begin{aligned}\binom{b}{i}\binom{c}{j}\binom{i}{k}\binom{j}{k} &= \frac{b!}{i!(b-i)!}\cdot\frac{c!}{j!(c-j)!}\cdot\frac{i!}{k!(i-k)!}\cdot\frac{j!}{k!(j-k)!}\\ &= \frac{b!}{k!(b-k)!}\cdot\frac{c!}{k!(c-k)!}\cdot\frac{(b-k)!}{(b-i)!(i-k)!}\cdot\frac{(c-k)!}{(c-j)!(j-k)!}\\ &= \binom{b}{k}\binom{c}{k}\binom{b-k}{b-i}\binom{c-k}{c-j}.\end{aligned}$$

Using Vandermonde's identity and the above identity, we have

$$\begin{aligned}\sum_{i+j=a}\binom{a}{i}\binom{b}{i}\binom{c}{j} &= \sum_{i+j=a}\binom{b}{i}\binom{c}{j}\sum_{k=0}^{a}\binom{i}{k}\binom{j}{k}\\ &= \sum_{k=0}^{a}\binom{b}{k}\binom{c}{k}\sum_{i+j=a}\binom{b-k}{b-i}\binom{c-k}{c-j}\\ &= \sum_{k=0}^{a}\binom{b}{k}\binom{c}{k}\binom{b+c-2k}{b+c-a}\\ &= \sum_{k=0}^{a}\binom{b}{k}\binom{c}{k}\binom{2a-2k}{a} = \sum_{i+j=a}\binom{b}{i}\binom{c}{i}\binom{2j}{a}.\end{aligned}$$

A cyclic inequality

H-813 Proposed by D. M. Bătineţu-Giurgiu, Bucharest, and Neculai Stanciu, Buzău, Romania (Vol. 55, No. 4, November 2017)

If $x_k > 0$ for $k = 1, \ldots, n$ and $m \geq 0$ is an integer, prove that

$$\left(\sum_{k=1}^{n}\frac{1}{x_k}\right)\sum_{\text{cyclic}}\frac{x_1x_2x_3}{L_mx_2x_3 + L_{m+1}x_3x_1 + L_{m+2}x_1x_2} \geq \frac{n^2}{2L_{m+2}}$$

and that the same inequality holds with the Lucas numbers replaced by the Fibonacci numbers.

Solution by Wei-Kai Lai and John Risher

We will prove that for positive A, B, C such that $A + B = C$, the inequality

$$\left(\sum_{k=1}^{n}\frac{1}{x_k}\right)\left(\sum_{\substack{i=1\\ cyclic}}^{n}\frac{x_ix_{i+1}x_{i+2}}{Ax_{i+1}x_{i+2} + Bx_{i+2}x_i + Cx_ix_{i+1}}\right) \geq \frac{n^2}{2C}$$

holds. According to the AM-HM inequality,

$$\begin{aligned}\sum_{\substack{i=1\\ cyclic}}^{n} \frac{x_i x_{i+1} x_{i+2}}{A x_{i+1} x_{i+2} + B x_{i+2} x_i + C x_i x_{i+1}} &\geq \frac{n^2}{\sum_{\substack{i=1\\ cyclic}}^{n} \left(\frac{A}{x_i} + \frac{B}{x_{i+1}} + \frac{C}{x_{i+2}}\right)} \\ &= \frac{n^2}{\left(\sum_{i=1}^{n} \frac{1}{x_i}\right)(A+B+C)} = \frac{n^2}{\left(\sum_{i=1}^{n} \frac{1}{x_i}\right)(2C)}.\end{aligned}$$

Therefore,

$$\left(\sum_{k=1}^{n} \frac{1}{x_k}\right)\left(\sum_{\substack{i=1\\ cyclic}}^{n} \frac{x_i x_{i+1} x_{i+2}}{A x_{i+1} x_{i+2} + B x_{i+2} x_i + C x_i x_{i+1}}\right) \geq \frac{n^2}{2C}$$

as claimed, thus proving the two inequalities in the original problem.

Also solved by Dmitry Fleischman and the proposers.

Errata: For Advanced Problem **H-838 (Vol. 57, No. 2, February 2019)** "$L_{n-(n+j)}$" should be "$L_{n-(r+j)}$". Furthermore, at the beginning of the published solution to Advanced Problem **H-809 (Vol. 57, No. 2, February 2019)** in the formulas for p_{2m} and p_{2m+1} in the right sides, the exponents of L should be "$2k-1$", "$2k$", and "$2m+1$" instead of "$k-1$", "k", and "$m+1$", respectively. The editor apologizes for these inconveniences.