ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
FLORIAN LUCA

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, SCHOOL OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG 3, WITS 2050, JOHANNESBURG, SOUTH AFRICA or by e-mail at the address florian.luca@wits.ac.za as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-858 Proposed by Muneer Jebreel Karama, Hebron, Palestine

Show that

$$\frac{1}{2}\left((2F_nF_{n+1})^8 + (F_{n-1}F_{n+2})^8 + F_{2n+1}^8\right)$$

is a perfect square for all $n \geq 0$.

H-859 Proposed by Robert Frontczak, Stuttgart, Germany

Prove that

$$\sum_{n\geq 1} \zeta(2n+1)\frac{F_{2n}}{5^n} = \frac{1}{2},$$

where $\zeta(k) = \sum_{n\geq 1} 1/n^k$ for $k \geq 2$ is the Riemann zeta function.

H-860 Proposed by Robert Frontczak, Stuttgart, Germany

Let $(B_n)_{n\geq 0}$ be the Bernoulli numbers defined by

$$\frac{z}{e^z-1} = \sum_{n\geq 0} B_n \frac{z^n}{n!} \quad (|z| < 2\pi).$$

Show that for all $n \geq 0$, we have

$$\sum_{\substack{k=0\\ k\equiv n \pmod 2}}^{n} \binom{n}{k}(2^kL_k - 2)5^{(n-k)/2}\frac{B_{n-k+2}}{n-k+2} = \frac{2^{n+2}L_{n+2}-2}{5(n+1)(n+2)} - 1.$$

H-861 Proposed by David Terr, Oceanside, CA

For arbitrary constants a, b, c, define the sequence $(G_n)_{n\geq 0}$ by $G_0 = a$, $G_1 = b$, $G_2 = c$, and the recurrence $G_n = G_{n-1} + G_{n-2} + G_{n-3}$ for $n \geq 3$. Find a closed form expression for

$$\sum_{j=0}^{n} G_{2j}G_{2j+1} \quad \text{valid for all} \quad n \geq 0.$$

H-862 Proposed by Ángel Plaza, Gran Canaria, Spain

Let $(F_{k,n})_{n\in\mathbb{Z}}$ and $(L_{k,n})_{n\in\mathbb{Z}}$ denote the k-Fibonacci and k-Lucas numbers given by $F_{k,n+1} = kF_{k,n} + F_{k,n-1}$, $L_{k,n+1} = kL_{k,n} + L_{k,n-1}$ for $n \geq 1$ with $F_{k,0} = 0$, $F_{k,1} = 1$, $L_{k,0} = 2$, $L_{k,1} = k$. Prove that for integers $m \geq 1$ and $j \geq 0$ we have

$$\text{(i)}\quad \sum_{n=1}^{m} F_{k,n\pm j}L_{k,n\mp j} = \frac{F_{k,2m+1}-1}{k} + \begin{cases} 0, & \text{if} \quad m \equiv 0 \pmod 2; \\ (-1)^j F_{k,\pm 2j}, & \text{if} \quad m \equiv 1 \pmod 2. \end{cases}$$

$$\text{(ii)}\quad \sum_{n=1}^{m} F_{k,n+j}F_{k,n-j}L_{k,n+j}L_{k,n-j} = \frac{F_{k,4m+2}/k - 1 - mL_{k,4j}}{k^2+4}.$$

SOLUTIONS

A circular inequality

H-825 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania (Vol. 56, No. 3, August 2018)

If $a, b, c > 0$ and n is a positive integer, prove that

$$2\left(\left(\frac{a}{bF_n + F_{n+1}c}\right)^3 + \left(\frac{b}{F_nc + F_{n+1}a}\right)^3 + \left(\frac{c}{F_na + F_{n+1}b}\right)^3\right)$$
$$+ \quad 3\frac{abc}{(F_na + F_{n+1}b)(F_nb + F_{n+1}c)(F_nc + F_{n+1}a)} \geq \frac{9}{F_{n+2}^3}.$$

Solution by Wei-Kai Lai, University of South Carolina Salkehatchie, Walterboro, SC

Let

$$x = \frac{a}{bF_n + F_{n+1}c}, \quad y = \frac{b}{cF_n + aF_{n+1}}, \quad z = \frac{c}{aF_n + bF_{n+1}}.$$

According to Surányi's inequality ([1], Theorem 4.4):

$$2\left(x^3 + y^3 + z^3\right) + 3xyz \geq (x + y + z)(x^2 + y^2 + z^2).$$

Because the quadratic mean is greater than or equal to the arithmetic mean, it is easy to check that

$$x^2 + y^2 + z^2 \geq \frac{1}{3}(x + y + z)^2.$$

So, we only need to prove that

$$\frac{1}{3}(x + y + z)^3 \geq \frac{9}{F_{n+2}^3},$$

or equivalently

$$x + y + z \geq \frac{3}{F_{n+2}}.$$

According to Radon's inequality,

$$\begin{aligned} x+y+z &= \frac{a^2}{F_nab+F_{n+1}ac}+\frac{b^2}{F_nbc+F_{n+1}ab}+\frac{c^2}{F_nac+F_{n+1}bc} \\ &\geq \frac{(a+b+c)^2}{(F_nab+F_{n+1}ac)+(F_nbc+F_{n+1}ab)+(F_nac+F_{n+1}bc)} \\ = & \frac{(a+b+c)^2}{(F_n+F_{n+1})(ab+ac+bc)} \geq \frac{3(ab+ac+bc)}{F_{n+2}(ab+ac+bc)}=\frac{3}{F_{n+2}}, \end{aligned}$$

hence proving the claimed inequality. The equality holds when $a=b=c$.

[1] Z. Cvetkovski, *Inequalities, Theorems, Techniques and Selected Problems*, Springer, New York, 2012, p. 35.

Also solved by Brian Bradie, Dmitry Fleischman, Ángel Plaza, Nicuşor Zlota, and the proposers.

Powers of 2 and powers of 3

H-826 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 56, No. 3, August 2018)

For an integer $n \geq 0$, prove that

$$\sum_{\substack{a+b=n \\ a,b\geq 0}} \frac{1}{L_{2^a3^b}F_{2^a3^{b+1}}} = \frac{F_{3^{n+1}-2^{n+1}}}{F_{3^{n+1}}F_{2^{n+1}}}.$$

Solution by the proposer

We use the identities

(1) $F_{s-t}=(-1)^t(F_{t+1}F_s-F_tF_{s+1})$ (see [1] (9));
(2) $F_{2s}=F_sL_s$ (see [1] (13)).

We have

$$\begin{aligned} \frac{F_{2^{a+1}3^b+1}}{F_{2^{a+1}3^b}}-\frac{F_{2^a3^{b+1}+1}}{F_{2^a3^{b+1}}} &= \frac{F_{2^{a+1}3^b+1}F_{2^a3^{b+1}}-F_{2^{a+1}3^b}F_{2^a3^{b+1}+1}}{F_{2^{a+1}3^b}F_{2^a3^{b+1}}} \\ = & \frac{F_{2^a3^{b+1}-2^{a+1}3^b}}{F_{2^{a+1}3^b}F_{2^a3^{b+1}}}(\text{by (1)}) = \frac{F_{2^a3^b}}{F_{2^{a+1}3^b}F_{2^a3^{b+1}}} = \frac{1}{L_{2^a3^b}F_{2^a3^{b+1}}} \text{ (by (2))}. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \sum_{\substack{a+b=n \\ a,b\geq 0}} \frac{1}{L_{2^a3^b}F_{2^a3^{b+1}}} &= \sum_{\substack{a+b=n \\ a,b\geq 0}} \left(\frac{F_{2^{a+1}3^b+1}}{F_{2^{a+1}3^b}}-\frac{F_{2^a3^{b+1}+1}}{F_{2^a3^{b+1}}}\right) \\ = & \sum_{a=0}^{n}\left(\frac{F_{2^{a+1}3^{n-a}+1}}{F_{2^{a+1}3^{n-a}}}-\frac{F_{2^a3^{n-a+1}+1}}{F_{2^a3^{n-a+1}}}\right) = \frac{F_{2^{n+1}+1}}{F_{2^{n+1}}}-\frac{F_{3^{n+1}+1}}{F_{3^{n+1}}} \\ = & \frac{F_{2^{n+1}+1}F_{3^{n+1}}-F_{2^{n+1}}F_{3^{n+1}+1}}{F_{2^{n+1}}F_{3^{n+1}}} = \frac{F_{3^{n+1}-2^{n+1}}}{F_{2^{n+1}}F_{3^{n+1}}} \text{ (by (1))}. \end{aligned}$$

[1] S. Vajda, *Fibonacci and Lucas numbers and the Golden Section*, Dover, 2008.

Also solved by Brian Bradie and Dmitry Fleischman.

A double limit

H-827 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania (Vol. 56, No. 3, August 2018)

Let $(a_n)_{n\geq 0}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} a_{n+1}/(na_n) = a > 0$. Compute

$$\lim_{m\to\infty}\left(\lim_{n\to\infty}\left(\left(\sqrt[n+1]{a_{n+1}}^{F_m/F_{m+1}} - (\sqrt[n]{a_n})^{F_m/F_{m+1}}\right) n^{F_{m-1}/F_{m+1}}\right)\right).$$

Solution by the proposers

Denoting $u_m = F_m/F_{m+1}$, we have $\lim_{m\to\infty} u_m = 1/\alpha$. We also have

$$\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{a_n}{n^n}} = \lim_{n\to\infty}\frac{a_{n+1}n^n}{a_n(n+1)^{n+1}} = \lim_{n\to\infty}\frac{a_{n+1}}{a_n n}\left(\frac{n}{n+1}\right)^{n+1} = \frac{a}{e}.$$

Denoting

$$v_n = \left(\frac{\sqrt[n+1]{a_{n+1}}}{\sqrt[n]{a_n}}\right)^{u_m},$$

we have $\lim_{n\to\infty} v_n = 1$, so that $\lim\limits_{n\to\infty}\dfrac{v_n-1}{\ln v_n} = 1$ and

$$\begin{aligned}\lim_{n\to\infty} v_n^n &= \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\frac{1}{\sqrt[n+1]{a_{n+1}}}\right)^{u_m} = \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n n}\frac{n+1}{\sqrt[n+1]{a_{n+1}}}\cdot\frac{n}{n+1}\right)^{u_m}\\ &= \left(a\cdot\frac{e}{a}\cdot 1\right)^{u_m} = e^{u_m}.\end{aligned}$$

Let

$$\begin{aligned}B_n(m) &= \left(\left(\sqrt[n+1]{a_{n+1}}\right)^{u_m} - \left(\sqrt[n]{a_n}\right)^{u_m}\right) n^{\frac{F_{m+1}-F_m}{F_{m+1}}}\\ &= \left(\sqrt[n]{a_n}\right)^{u_m}(v_n-1)n^{1-u_m}\\ &= \left(\frac{\sqrt[n]{a_n}}{n}\right)^{u_m}\frac{v_n-1}{\ln v_n}\cdot n\cdot\ln v_n\\ &= \left(\frac{\sqrt[n]{a_n}}{n}\right)^{u_m}\frac{v_n-1}{\ln v_n}\ln v_n^n.\end{aligned}$$

Hence, the limit to compute is

$$\lim_{m\to\infty}\lim_{n\to\infty} B_n(m) = \lim_{m\to\infty}\left(\left(\frac{a}{e}\right)^{u_m}\cdot 1\cdot\ln e^{u_m}\right) = \left(\frac{a}{e}\right)^{1/\alpha}\frac{1}{\alpha}.$$

Also solved by Dmitry Fleischman.

Closed formula for the weighted sum of squares of Tribonacci numbers

H-828 Proposed by Kenneth Davenport, Dallas, PA (Vol. 56, No. 3, August 2018)

Find a closed form expression for

$$\sum_{k=0}^{n} kT_k^2,$$

where $(T_k)_{k\geq 0}$ is the sequence of Tribonacci numbers satisfying $T_0 = 0$, $T_1 = T_2 = 1$, and $T_{k+3} = T_{k+2} + T_{k+1} + T_k$ for all $k \geq 0$.

Solution by Raphael Schumacher, ETH, Zurich, Switzerland

We will prove that

$$\sum_{k=0}^{n} kT_k^2 = \left(\frac{1}{2}n+1\right)T_nT_{n+2} + (n+2)\,T_{n+1}T_{n+2} - \left(\frac{1}{4}n+1\right)T_n^2 - \left(n+\frac{7}{4}\right)T_{n+1}^2 - \left(\frac{1}{4}n+\frac{3}{4}\right)T_{n+2}^2 - \frac{1}{2}T_nT_{n+1} + \frac{1}{2}. \tag{1}$$

We have that (see [1])

$$T_n = \frac{\alpha_1^n}{4\alpha_1-\alpha_1^2-1} + \frac{\alpha_2^n}{4\alpha_2-\alpha_2^2-1} + \frac{\alpha_3^n}{4\alpha_3-\alpha_3^2-1} \quad n \geq 0,$$

where α_1, α_2, and α_3 are the three roots of the polynomial $p(x) := x^3-x^2-x-1$. Using the explicit formula for the Tribonacci numbers from above and the two infinite series identities

$$\sum_{k=0}^{\infty} y^k = \frac{1}{1-y} \quad \text{and} \quad \sum_{k=0}^{\infty} ky^k = \frac{y}{(1-y)^2} \quad \text{for all } y \in \mathbb{C} \text{ with } |y|<1,$$

we get the following generating function identities

$$f_1(x) := \sum_{n=0}^{\infty} T_n^2 x^n = -\frac{x\left(x^3+x^2+x-1\right)}{(x^3-x^2-x-1)(x^3+x^2+3x-1)},$$

$$f_2(x) := \sum_{n=0}^{\infty} T_{n+1}^2 x^n = -\frac{x^3+x^2+x-1}{(x^3-x^2-x-1)(x^3+x^2+3x-1)},$$

$$f_3(x) := \sum_{n=0}^{\infty} T_{n+2}^2 x^n = -\frac{x^5+x^3-5x^2-2x-1}{(x^3-x^2-x-1)(x^3+x^2+3x-1)},$$

$$f_4(x) := \sum_{n=0}^{\infty} T_nT_{n+1} x^n = \frac{x\left(x^2+1\right)}{(x^3-x^2-x-1)(x^3+x^2+3x-1)},$$

$$f_5(x) := \sum_{n=0}^{\infty} T_nT_{n+2} x^n = \frac{2x}{(x^3-x^2-x-1)(x^3+x^2+3x-1)},$$

$$f_6(x) := \sum_{n=0}^{\infty} T_{n+1}T_{n+2} x^n = \frac{x^2+1}{(x^3-x^2-x-1)(x^3+x^2+3x-1)},$$

$$f_7(x) := \sum_{n=0}^{\infty} nT_n^2 x^n = \frac{x\left(2x^9+3x^8+4x^7+2x^6+8x^5+12x^3+2x^2-2x+1\right)}{(x^3-x^2-x-1)^2(x^3+x^2+3x-1)^2},$$

$$f_8(x) := \sum_{n=0}^{\infty} nT_{n+1}^2 x^n = \frac{x\left(3x^8+4x^7+6x^6-4x^5-12x^3+14x^2+4x+1\right)}{(x^3-x^2-x-1)^2(x^3+x^2+3x-1)^2},$$

$$f_9(x) := \sum_{n=0}^{\infty} nT_{n+2}^2 x^n = \frac{x\left(x^{10}+2x^8-8x^7-8x^5+22x^4+24x^3+11x^2+16x+4\right)}{(x^3-x^2-x-1)^2(x^3+x^2+3x-1)^2},$$

$$f_{10}(x) := \sum_{n=0}^{\infty} nT_nT_{n+2} x^n = -\frac{2x\left(5x^6+3x^4-12x^3-3x^2-1\right)}{(x^3-x^2-x-1)^2(x^3+x^2+3x-1)^2},$$

$$f_{11}(x) := \sum_{n=0}^{\infty} nT_{n+1}T_{n+2}x^n = -\frac{2x\left(2x^7+4x^5-3x^4+2x^3-8x^2-4x-1\right)}{\left(x^3-x^2-x-1\right)^2\left(x^3+x^2+3x-1\right)^2},$$
$$f_{12}(x) := \sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

as well as

$$\sum_{k=0}^{\infty} kT_k^2x^k = \frac{x\left(2x^9+3x^8+4x^7+2x^6+8x^5+12x^3+2x^2-2x+1\right)}{\left(x^3-x^2-x-1\right)^2\left(x^3+x^2+3x-1\right)^2},$$
$$\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} kT_k^2\right)x^n = \frac{1}{1-x}\sum_{k=0}^{\infty} kT_k^2x^k = \frac{x\left(2x^9+3x^8+4x^7+2x^6+8x^5+12x^3+2x^2-2x+1\right)}{(1-x)\left(x^3-x^2-x-1\right)^2\left(x^3+x^2+3x-1\right)^2}.$$

Because of the calculation

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} kT_k^2\right)x^n = \frac{x\left(2x^9+3x^8+4x^7+2x^6+8x^5+12x^3+2x^2-2x+1\right)}{(1-x)\left(x^3-x^2-x-1\right)^2\left(x^3+x^2+3x-1\right)^2}\\
&= f_{12}(x)f_7(x)\\
&= \frac{1}{2}f_{10}(x)+f_5(x)+f_{11}(x)+2f_6(x)-\frac{1}{4}f_7(x)-f_1(x)-f_8(x)-\frac{7}{4}f_2(x)-\frac{1}{4}f_9(x)\\
&\quad -\frac{3}{4}f_3(x)-\frac{1}{2}f_4(x)+\frac{1}{2}f_{12}(x)\\
&= \frac{1}{2}\sum_{n=0}^{\infty} nT_nT_{n+2}x^n+\sum_{n=0}^{\infty} T_nT_{n+2}x^n+\sum_{n=0}^{\infty} nT_{n+1}T_{n+2}x^n+2\sum_{n=0}^{\infty} T_{n+1}T_{n+2}x^n\\
&\quad -\frac{1}{4}\sum_{n=0}^{\infty} nT_n^2x^n-\sum_{n=0}^{\infty} T_n^2x^n-\sum_{n=0}^{\infty} nT_{n+1}^2x^n-\frac{7}{4}\sum_{n=0}^{\infty} T_{n+1}^2x^n-\frac{1}{4}\sum_{n=0}^{\infty} nT_{n+2}^2x^n\\
&\quad -\frac{3}{4}\sum_{n=0}^{\infty} T_{n+2}^2x^n-\frac{1}{2}\sum_{n=0}^{\infty} T_nT_{n+1}x^n+\frac{1}{2}\sum_{n=0}^{\infty} x^n\\
&= \sum_{n=0}^{\infty}\Bigg[\left(\frac{1}{2}n+1\right)T_nT_{n+2}+(n+2)\,T_{n+1}T_{n+2}-\left(\frac{1}{4}n+1\right)T_n^2-\left(n+\frac{7}{4}\right)T_{n+1}^2\\
&\quad -\left(\frac{1}{4}n+\frac{3}{4}\right)T_{n+2}^2-\frac{1}{2}T_nT_{n+1}+\frac{1}{2}\Bigg]x^n,
\end{aligned}$$

the formula

$$\begin{aligned}
\sum_{k=0}^{n} kT_k^2 = {}&\left(\frac{1}{2}n+1\right)T_nT_{n+2}+(n+2)\,T_{n+1}T_{n+2}-\left(\frac{1}{4}n+1\right)T_n^2-\left(n+\frac{7}{4}\right)T_{n+1}^2\\
&-\left(\frac{1}{4}n+\frac{3}{4}\right)T_{n+2}^2-\frac{1}{2}T_nT_{n+1}+\frac{1}{2}
\end{aligned}$$

holds true by equating coefficients and using the uniqueness of power series expansions. This expression is (1).

Substituting the relation $T_n := T_{n+3} - T_{n+2} - T_{n+1}$ into the equation (1), we get the equivalent form

$$\sum_{k=0}^{n} kT_k^2 = \frac{1}{2}(n+3)\,T_{n+1}T_{n+3} + (n+3)\,T_{n+2}T_{n+3} - \left(\frac{5}{4}n + \frac{9}{4}\right)T_{n+1}^2 - \left(n + \frac{11}{4}\right)T_{n+2}^2 - \left(\frac{1}{4}n + 1\right)T_{n+3}^2 - \frac{1}{2}T_{n+1}T_{n+2} + \frac{1}{2}. \tag{2}$$

[1] http://mathworld.wolfram.com/TribonacciNumber.html

Also solved by Dmitry Fleischman, Jason Smith, David Terr, and the proposer.

Retraction: The popular problem **H-856** was initially rejected by this Department and ended up appearing in print by an editorial mishap. Afterwards part 1) involving the Fibonacci numbers was submitted and meanwhile appeared as Problem 4517, Crux Mathematicorum **46**, no. 2, February 2020. This Department apologizes for this mishap and wishes to retract the Fibonacci part of Problem **H-856**. The part involving Lucas numbers is retained.