ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit solutions and problem proposals to Dr. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY, 14063, or by email at kwong@fredonia.edu. If you wish to have receipt of your submission acknowledged by mail, please include a self-addressed, stamped envelope.

Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by February 15, 2018. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results."

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1194 (Corrected) Proposed by D. M. Bătineţu-Giurgui, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School, Buzău, Romania.

Prove that

$$\frac{L_1}{(L_1^2+L_2^2+2)^{m+1}} + \frac{L_2}{(L_1^2+L_2^2+L_3^2+2)^{m+1}} + \cdots + \frac{L_n}{(L_1^2+L_2^2+\cdots+L_{n+1}^2+2)^{m+1}} \geq \frac{L_{n+2}-3}{(3L_{n+2})^{m+1}}$$

for any positive integers n and m.

B-1211 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

For $n \geq 1$, prove that

$$F_{n-1}^3 + \sum_{k=1}^{n} F_k^3 = \frac{F_{3n-1}+1}{2}.$$

B-1212 **Proposed by D. M. Bătineţu-Giurgui, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School, Buzău, Romania.**

Prove that

$$\frac{F_n^4+1}{F_n^2-F_n+1}+\sum_{k=1}^{n-1}\frac{F_k^4+F_{k+1}^4}{F_k^2-F_kF_{k+1}+F_{k+1}^2}>2F_nF_{n+1}$$

for any positive integer n.

B-1213 **Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.**

For every positive integer n, prove that

$$\frac{F_1}{F_3}\cdot\frac{F_5}{F_7}\cdot\ \cdots\ \cdot\frac{F_{4n-3}}{F_{4n-1}}>\sqrt[4]{\frac{1}{F_1+F_5+\cdots+F_{8n+1}}},$$

and

$$\frac{F_2}{F_4}\cdot\frac{F_6}{F_8}\cdot\ \cdots\ \cdot\frac{F_{4n-2}}{F_{4n}}<\sqrt[4]{\frac{2}{F_3+F_7+\cdots+F_{8n+3}}}.$$

B-1214 **Proposed by Hideyuki Ohtsuka, Saitama, Japan.**

Given an integer $m\geq 2$, find a closed form for the infinite sum

$$\sum_{n=1}^{\infty}\frac{F_{2n+m}}{F_nF_{n+2}F_{n+m-2}F_{n+m}}.$$

B-1215 **Proposed by Ángel Plaza and Sergio Falcón, Universidad de Las Palmas de Gran Canaria, Spain.**

For any positive integer k, the k-Fibonacci and k-Lucas sequences $\{F_{k,n}\}_{n\in\mathbb{N}}$ and $\{L_{k,n}\}_{n\in\mathbb{N}}$ are defined recursively by $u_{n+1}=ku_n+u_{n-1}$ for $n\geq 1$, with respective initial conditions $F_{k,0}=0$, $F_{k,1}=1$, and $L_{k=0}=2$, $L_{k,1}=k$. Let c be a positive integer. The sequence $\{a_n\}_{n\in\mathbb{N}}$ is defined by $a_1=1$, $a_2=3$, and $a_{n+2}=a_n+2c$ for $n\geq 1$. Prove that

$$\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{F_{k,c}}{F_{k,a_n+c}}\right)=\tan^{-1}\left(\frac{1}{k}\right),\qquad \text{if } c \text{ is even;}$$

$$\sum_{n=1}^{\infty}\tan^{-1}\left(\frac{L_{k,c}}{L_{k,a_n+c}}\right)=\tan^{-1}\left(\frac{1}{k}\right),\qquad \text{if } c \text{ is odd.}$$

SOLUTIONS

Binet! Binet! Binet!

B-1191 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 54.3, August 2016)

For nonnegative integers m and n, prove that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{L_{mk}}{L_m^k}=\frac{L_{mn}}{L_m^n}.$$

Solution by Jason L. Smith, Richland Community College, Decatur, IL.

The fraction in the summand can be written as

$$\frac{L_{mk}}{L_m^k}=\frac{\alpha^{mk}+\beta^{mk}}{L_m^k}=\left(\frac{\alpha^m}{L_m}\right)^k+\left(\frac{\beta^m}{L_m}\right)^k.$$

The binomial theorem can be applied thus:

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{L_{mk}}{L_m^k}&=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left[\left(\frac{\alpha^m}{L_m}\right)^k+\left(\frac{\beta^m}{L_m}\right)^k\right]\\&=\left(1-\frac{\alpha^m}{L_m}\right)^n+\left(1-\frac{\beta^m}{L_m}\right)^n\\&=\left(\frac{\beta^m}{L_m}\right)^n+\left(\frac{\alpha^m}{L_m}\right)^n\\&=\frac{\beta^{mn}+\alpha^{mn}}{L_m^n}\\&=\frac{L_{mn}}{L_m^n}.\end{aligned}$$

Also solved by Brian Bradie, Kenny B. Davenport, Steve Edwards, I. V. Fedak, Dmitry Fleischman, Ángel Plaza, Hemlatha Rajpurohit, David Terr, and the proposer.

Lower Hessenberg Matrix

B-1192 Proposed by T. Goy, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.
(Vol. 54.3, August 2016)

Let M_n be an $n\times n$ matrix given for all $n\geq 1$ by

$$M_n=\begin{pmatrix}F_1 & 1 & 0 & \dots & 0 & 0 & 0\\ F_2 & F_1 & 1 & \dots & 0 & 0 & 0\\ F_3 & F_2 & F_1 & \dots & 0 & 0 & 0\\ \dots & \dots & \dots & \ddots & \dots & \dots & \dots\\ F_{n-1} & F_{n-2} & F_{n-3} & \dots & F_2 & F_1 & 1\\ F_n & F_{n-1} & F_{n-2} & \dots & F_3 & F_2 & F_1\end{pmatrix}.$$

Prove that

$$\det(M_n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

Solution by Brian Bradie, Christopher Newport University, Newport, VA.

Note that

$$M_1 = (F_1) \qquad \text{and} \qquad M_2 = \begin{pmatrix} F_1 & 1 \\ F_1 & F_1 \end{pmatrix},$$

so that $\det(M_1) = 1$ and $\det(M_2) = 0$. For $n \geq 3$, subtracting the second column from the first column of M_n leads to

$$\det(M_n) = \begin{vmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & F_1 & 1 & \dots & 0 & 0 & 0 \\ F_1 & F_2 & F_1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \ddots & \dots & \dots & \dots \\ F_{n-3} & F_{n-2} & F_{n-3} & \dots & F_2 & F_1 & 1 \\ F_{n-2} & F_{n-1} & F_{n-2} & \dots & F_3 & F_2 & F_1 \end{vmatrix}.$$

Subtraction of the third column from the second column further reduces the determinant to

$$\det(M_n) = \begin{vmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ F_1 & 0 & F_1 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \ddots & \dots & \dots & \dots \\ F_{n-3} & F_{n-4} & F_{n-3} & \dots & F_2 & F_1 & 1 \\ F_{n-2} & F_{n-3} & F_{n-2} & \dots & F_3 & F_2 & F_1 \end{vmatrix} = \det(M_{n-2}).$$

The recurrence relation $\det(M_n) = \det(M_{n-2})$ together with $\det(M_1) = 1$ and $\det(M_2) = 0$ yields

$$\det(M_n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

Editor's Notes. Kuhapatanskul and Plaza (independently) remarked that the matrix is a lower Hessenberg matrix, whose determinant is given in [1]. Kenny B. Davenport commented that this problem is a special case of a result in [2].

References

[1] N. D. Cahill, J. R. D'Errico, D. A. Narayan, and J. Y. Narayan, Fibonacci determinants, *College Math. J.*, **33** (2002), 221–225.

[2] A. J. Macfarlan, *Use of determinants to present identities involving Fibonacci and related numbers*, The Fibonacci Quarterly, **48** (2010), 68–76.

Also solved by Jeremiah Bartz, Hsin-Yun Ching (student), Steve Edwards, I. V. Fedak, Dmitry Fleischman, Kantaphon Kuhapatanakul, Mithun Kumar das, Ángel Plaza, David Stone and John Hawkins (jointly), Dan Weiner, and the proposer.

Picking the Right Numbers

B-1193 Proposed by José Luis Diaz-Barrero, Barcelona Tech, Barcelona, Spain. (Vol. 54.3, August 2016)

If $F_1^2, F_2^2, \ldots, F_n^2$ are the square of the first n Fibonacci numbers, then find real numbers $a_1, a_2, \ldots, a_n$ satisfying $a_k > F_k^2$, $1 \le k \le n$, and

$$\frac{1}{F_n F_{n+1}} \sum_{k=1}^{n} a_k < \frac{\alpha^2}{\alpha - 1}.$$

Solution by Steve Edwards, Kennesaw State University, Marietta, GA.

Since $\frac{\alpha^2}{\alpha-1} = \frac{\alpha+1}{\alpha-1} > 1$, we can let a_k be any real number satisfying $F_k^2 < a_k < \frac{\alpha^2}{\alpha-1} F_k^2$. Then, since $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$, we find

$$\frac{1}{F_n F_{n+1}} \sum_{k=1}^{n} a_k < \frac{1}{F_n F_{n+1}} \sum_{k=1}^{n} \frac{\alpha^2}{\alpha - 1} F_k^2 = \frac{1}{F_n F_{n+1}} \cdot \frac{\alpha^2}{\alpha - 1} F_n F_{n+1} = \frac{\alpha^2}{\alpha - 1}.$$

Also solved by Brian D. Beasley, Brian Bradie, Kenny B. Davenport, I. V. Fedak, Dmitry G. Fleishcman, Ángel Plaza, David Stone and John Hawkins (jointly), and the proposer.

Errors!

B-1194 Proposed by D. M. Bătineţu-Giurgui, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School, Buzău, Romania. (Vol. 54.3, August 2016)

Prove that

$$\frac{L_1}{(L_1^2 + L_2^2 + 2)^{m+1}} + \frac{L_2}{(L_1^2 + L_2^2 + L_3^2 + 2)^{m+1}} + \cdots + \frac{L_n}{(L_1^2 + L_2^2 + \cdots + L_{n+1}^2 + 2)^{m+1}}$$
$$\ge \frac{(L_{n+2} - 1)^{m+1}}{L_{n+2}^{m+1}(L_{n+2} - 3)^m}.$$

for any positive integers n and m.

Editor's Remark. The right-hand side of the inequality was incorrectly stated in the original problem. The correct version can be found at the beginning of the section in this issue.

Polygon with Generalized Fibonacci Numbers as Its Vertices

B-1195 Proposed by Jeremiah Bartz, Francis Marion University, Florence, SC. (Vol. 54.3, August 2016)

Let G_i denote the generalized Fibonacci sequence given by $G_0 = a$, $G_1 = b$, and $G_i = G_{i-1} + G_{i-2}$ for $i \ge 3$. Let $m \ge 0$ and $k \ge 0$. Prove that the area A of the polygon with $n \ge 3$ vertices

$$(G_m, G_{m+k}), (G_{m+2k}, G_{m+3k}), \ldots, (G_{m+(2n-2)k}, G_{m+(2n-1)k})$$

is

$$\frac{|\mu| F_k \big(F_{2k(n-1)} - (n-1)F_{2k}\big)}{2}$$

where $\mu = a^2 + ab - b^2$.

Solution by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

It is known [1, Theorem 33.3] that the area with vertices (G_n, G_{n+r}), (G_{n+p}, G_{n+p+r}), and (G_{n+q}, G_{n+q+r}) is independent of n, and equals

$$\frac{1}{2}\left|\mu F_r\big((-1)^p F_{q-p} + F_p - F_q\big)\right|.$$

For $n = m$, $r = k$, $p = 2ks$, where $1 \le s \le n-2$, and $q = p + 2k$, we obtain the area

$$\frac{1}{2}\left|\mu F_k\big(F_{2k} + F_{2ks} - F_{2k(s+1)}\big)\right| = \frac{1}{2}\,|\mu| F_k\big(F_{2k(s+1)} - F_{2ks} - F_{2k}\big).$$

Therefore,

$$\begin{aligned} A &= \frac{1}{2}\,|\mu| F_k \sum_{s=1}^{n-2} \big(F_{2k(s+1)} - F_{2ks} - F_{2k}\big) \\ &= \frac{1}{2}\,|\mu| F_k \big(F_{2k(n-1)} - F_{2k} - (n-2)F_{2k}\big) \\ &= \frac{1}{2}\,|\mu| F_k \big(F_{2k(n-1)} - (n-1)F_{2k}\big). \end{aligned}$$

References

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley, New York, 2001.

Also solved by the proposer.