ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
RUSS EULER AND JAWAD SADEK

Please submit all new problem proposals and their solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468, or by email at reuler@nwmissouri.edu. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by August 15, 2015. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2,\ \beta = (1-\sqrt{5})/2,\ F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1161 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Prove each of the following:

(i) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k L_{2^k}}{F_{2^k}^2} = 10,$

(ii) $\displaystyle\sum_{k=0}^{\infty} \tan^{-1}\left(\frac{1}{\sqrt{5}F_{2k+1}}\right) = \frac{\pi}{4}.$

B-1162 **Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.**

Let n be a positive integer. Show that

$$\sum_{k=1}^{n} \sqrt{\binom{n-1}{k-1}\frac{F_k}{k}} \le \sqrt{F_{2n}}\ .$$

B-1163 **Proposed by Ángel Plaza and Sergio Falcón, Universidad de Las Palmas de Gran Canaria, Spain.**

For any positive integer k, the k-Fibonacci and the k-Lucas sequences, $\{F_{k,n}\}_{n\in\mathbb{N}}$ and $\{L_{k,n}\}_{n\in\mathbb{N}}$, both are defined recursively by $u_{n+1} = ku_n + u_{n-1}$ for $n \ge 1$ with respective initial conditions $F_{k,0} = 0$, $F_{k,1} = 1$, and $L_{k,0} = 2$, $L_{k,1} = k$. For any integer $n \ge 2$, prove that

(i) $$\sum_{j=1}^{n}\left(\frac{kF_{k,j}}{F_{k,n+1}+F_{k,n}-1-kF_{k,j}}\right)^2 \ge \frac{n}{(n-1)^2},$$

(ii) $$\sum_{j=1}^{n}\left(\frac{kL_{k,j}}{L_{k,n+1}+L_{k,n}-2-k-kL_{k,j}}\right)^2 \ge \frac{n}{(n-1)^2}.$$

B-1164 **Proposed by Hideyuki Ohtsuka, Saitama, Japan.**

Determine each of the following:

(i) $$\sum_{n=0}^{\infty}\frac{(-1)^n}{L_{2F_n}L_{2F_{n+3}}}$$

(ii) $$\sum_{n=0}^{\infty}\frac{(-1)^n}{L_{2L_n}L_{2L_{n+3}}}.$$

B-1165 **Proposed by Hideyuki Ohtsuka, Saitama, Japan.**

For an integer $n \neq 0$, find the value of

$$\frac{L_{F_{3n}}}{F_{F_{3n-1}}F_{F_{3n-2}}} + \frac{L_{F_{3n-1}}}{F_{F_{3n-2}}F_{F_{3n}}} + \frac{L_{F_{3n-2}}}{F_{F_{3n}}F_{F_{3n-1}}}.$$

SOLUTIONS

Easily Seen by "Telescoping"

B-1141 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 52.1, February 2014)

Determine

$$\sum_{k=1}^{\infty} \frac{2^k \sin(2^k\theta)}{L_{2^k} + 2\cos(2^k\theta)}.$$

Solution by Kenneth B. Davenport, DA, PA.

We will use the following identities:

(i) $\sin 2x = 2\sin x \cos x$.
(ii) $\cos 2x = 1 - 2\sin^2 x = 2\cos^2 x - 1$.
(iii) $L_n^2 = L_{2n} + 2(-1)^n$.

The key to our solution is to write the sum as a "telescoping" sum. Note that

$$\frac{1}{L_{2^k} - 2\cos 2^k\theta} - \frac{1}{L_{2^k} + 2\cos 2^k\theta} = \frac{4\cos 2^k\theta}{L_{2^k}^2 - 4\cos^2(2^k\theta)}. \tag{1}$$

Based on (ii) and (iii), we may write the denominator on the right-hand side as

$$(L_{2^{k+1}} - 2\cos 2^{k+1}\theta).$$

So now (1) is written:

$$\frac{1}{L_{2^k} + 2\cos 2^k\theta} = \frac{1}{L_{2^k} - 2\cos 2^k\theta} - \frac{4\cos 2^k\theta}{L_{2^{k+1}} - 2\cos 2^{k+1}\theta}.$$

Next, by multiplying both sides by $2^k \sin 2^k\theta$ and then summing we get

$$\sum_{k=1}^{N} \frac{2^k \sin 2^k\theta}{L_{2^k} + \cos 2^k\theta} = \sum_{k=1}^{N} \left(\frac{2^k \sin 2^k\theta}{L_{2^k} - 2\cos 2^k\theta} - \frac{4 \cdot 2^k \sin 2^k\theta \cos 2^k\theta}{L_{2^{k+1}} - 2\cos 2^{k+1}\theta} \right). \tag{2}$$

Using (i), the second term's numerator on the right-hand side is now written $2^{k+1}\sin 2^{k+1}\theta$. Consequently, we now have the desired collapsing sum we were seeking, i.e.

$$\frac{2\sin 2\theta}{L_2 - 2\cos 2\theta} - \frac{2^{N+1}\sin 2^{N+1}\theta}{L_2^{N+1} - 2\cos 2^{N+1}\theta}. \tag{3}$$

Now let $N \to \infty$. From the Binet form of the Lucas numbers, and the fact that $\alpha^2 > 2$, it is easy to see that the second term in (3) vanishes quickly. Hence, we have shown

$$\sum_{k=1}^{\infty} \frac{2^k \sin 2^k\theta}{L_{2^k} + 2\cos 2^k\theta} = \frac{2\sin 2\theta}{3 - 2\cos 2\theta}.$$

Also solved by the proposer.

Summing Every Fourth Fibonacci

B-1142 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania. (Vol. 52.1, February 2014)

Prove that $\sum_{k=1}^{n} F_{4k-1} = F_{2n} \cdot F_{2n+1}$ for any positive integer n.

Solution by Carlos Alirio Rico Acevedo, Universidad Distrital Francisco José de Caldas (ITENU), Bogotá, Columbia.

Proof. We know that $F_{2n+1} = F_n^2 + F_{n+1}^2$ and $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$, see, for example [1, p. 79,Corollary 5.4] and [1, p. 77, Theorem 5.5], respectively. Thus,

$$\begin{aligned} \sum_{k=1}^{n} F_{4k-1} &= \sum_{k=1}^{n} (F_{2k}^2 + F_{2k-1}^2) \\ &= \sum_{k=1}^{2n} F_k^2 \\ &= F_{2n} F_{2n+1}. \end{aligned}$$

This proves the result.

References

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley, New York, 2001.

All solutions were either some variations on the featured solution or they used an induction argument.

Also solved by Brian D. Beasley, Charles K. Cook, Kenneth B. Davenport (2 solutions), Steve Edwards, Bryan Ek, Pedro Fernando Fernández Espiroza (student), Ralph Grimaldi (2 solutions), Russell Jay Hendel, Tia Herring, Gary Knight, Shane Latchman, Ludwing Murillo, Zibussion Ndimande, and Eyob Tarekegn (jointly), Harris Kwong (2 solutions), Wei-kai Lai, Fidel Ngwane and Gregory Jay (student) (jointly), Kathleen E. Lewis, Michelle Monnin, Ángel Plaza, Hideyuki Ohtsuka, Cecil Rousseau, Jason L. Smith, Lawrence Sommer, David Stone and John Hawkins (jointly), Dan Weiner, Nazmiye Yilmaz, and the proposer. A nameless solution was also received.

Just Apply the AM-GM!

B-1143 Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain and Francesc Gispert Sánchez, CFIS, Barcelona Tech, Barcelona, Spain. (Vol. 52.1, February 2014)

Let n be a positive integer. Prove that

$$\frac{1}{F_nF_{n+1}}\left[\left(1-\frac{1}{n}\right)\sum_{k=1}^{n}F_k^{2n}+\prod_{k=1}^{n}F_k^2\right]\geq\left(\prod_{k=1}^{n}F_k^{(1-1/n)}\right)^2.$$

Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

Using $F_nF_{n+1}=\sum_{k=1}^{n}F_k^2$, the proposed inequality may be written equivalently as

$$(n-1)\frac{\sum_{k=1}^{n}F_k^{2n}}{n}+\prod_{k=1}^{n}F_k^2\geq\left(\prod_{k=1}^{n}F_k^2\right)^{(n-1)/n}\sum_{k=1}^{n}F_k^2,$$

which follows by the AM-GM inequality.

Also solved by Dmitry Fleischman and the proposer.

It Is Not Strict When $n=1$!

B-1144 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania. (Vol. 52.1, February 2014)

Prove that

$$\prod_{k=1}^{n}(F_k^2+1)>F_n\cdot F_{n+1}+1 \tag{1}$$

$$\prod_{k=1}^{n}(L_k^2+1)>L_n\cdot L_{n+1}-1 \tag{2}$$

for any positive integer n.

Solution 1 by Harris Kwong, SUNY Fredonia, Fredonia, NY.

Both inequalities actually become equations when $n=1$, so we may assume $n>1$. Then each product contains at least two factors. Hence, since F_k^2, $L_k^2>0$, upon expansion, we find

$$\prod_{k=1}^{n}(F_k^2+1)>1+\sum_{k=1}^{n}F_k^2=1+F_nF_{n+1},$$

and

$$\prod_{k=1}^{n}(L_k^2+1)>1+\sum_{k=1}^{n}L_k^2=1+(L_nL_{n+1}-2)=L_nL_{n+1}-1.$$

Solution 2 by Sydney Marks and Leah Seader (jointly) students at California University of Pennsylvania (CALURMA), California, PA.

Proof. We generalized inequalities (1) and (2) by proving

$$\prod_{k=1}^{n}(G_k^2+1) \geq G_nG_{n+1}+a(a-b)+1 \quad \text{for all } n \in \mathbb{N}, \tag{3}$$

where $\{G_n\}_{n\in\mathbb{N}}$ is the generalized Fibonacci sequence with $G_1 = a$ and $G_2 = b$, where $a, b \in \mathbb{Z}$. We first notice that

$$\prod_{k=1}^{n}(G_k^2+1) \geq \left(\sum_{k=1}^{n} G_k^2\right)+1. \tag{4}$$

We prove inequality (4) by induction on n. This inequality is clearly true when $n = 1$. We assume that (4) is true for a fixed arbitrary natural number n. Thus,

$$\begin{aligned}\prod_{k=1}^{n+1}(G_k^2+1) = \left(\prod_{k=1}^{n}(G_k^2+1)\right)(G_{n+1}^2+1) &\geq \left[\left(\sum_{k=1}^{n} G_k^2\right)+1\right](G_{n+1}^2+1)\\ &= G_{n+1}^2\left(\sum_{k=1}^{n} G_k^2\right)+\left(\sum_{k=1}^{n} G_k^2\right)+G_{n+1}^2+1\\ &\geq \left(\sum_{k=1}^{n+1} G_k^2\right)+1.\end{aligned}$$

Therefore, (4) is true for every $n \in \mathbb{N}$. Thus, inequality (3) follows from (4) by using $\sum_{n=1}^{n} G_k^2 = G_nG_{n+1}+a(a-b)$, see (1, Exercise 14, p. 113]. Inequalities (1) and (2) are now obtained from (3) by setting $a = b = 1$ and $a = 1$, $b = 3$, respectively.

References

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley, New York, 2001.

Also solved by Brian D. Beasley, Charles K. Cook, Kenneth B. Davenport, Steve Edwards, Dmitry Fleischman, Ralph Grimaldi, Russell Jay Hendel, Tia Herring, Gary Knight, Shane Latchman, Ludwig Murillo, Zibussion Ndimande and Eyob Tarekegn (jointly) (students), Wei-kai Lai, Kathleen E. Lewis, Carolina Melee Lopez (student), Hideyuki Ohtsuka, Ángel Plaza, David Stone and John Hawkins (jointly), and the proposer.

Squares of "Almost Squares" Terms

B-1145 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania. (Vol. 52.1, February 2014)

Prove that

$$\left(F_1-\sqrt{F_1F_2}+F_2\right)^2+\left(F_2-\sqrt{F_2F_3}+F_3\right)^2+\cdots+\left(F_n-\sqrt{F_nF_1}+F_1\right)^2 \geq F_nF_{n+1} \quad (1)$$

$$\left(L_1 - \sqrt{L_1L_2} + L_2\right)^2 + \left(L_2 - \sqrt{L_2L_3} + L_3\right)^2 + \cdots + \left(L_n - \sqrt{L_nL_1} + L_1\right)^2 \geq L_nL_{n+1} - 2 \quad (2)$$

for any positive integer n.

Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaris, Spain.

Using the AM-GM inequality, we obtain

$$\frac{x+y}{2} \geq \frac{\sqrt{xy} + \sqrt{\frac{x^2+y^2}{2}}}{2},$$

and so

$$x - \sqrt{xy} + y \geq \sqrt{\frac{x^2+y^2}{2}}$$

for any positive real numbers x and y. Thus, the left-hand side of (1), *LHS*, is

$$\begin{aligned} LHS &\geq \frac{F_1^2+F_2^2}{2} + \frac{F_2^2+F_3^2}{2} + \cdots + \frac{F_n^2+F_1^2}{2} \\ &= \sum_{k=1}^{n} F_k^2 \\ &= F_nF_{n+1}. \end{aligned}$$

Inequality (2) is proved in the same way by using $\sum_{k=1}^{n} L_k^2 = L_nL_{n+1} - 2$.

Also solved by Kenneth B. Davenport, Dmitry Fleischman, Russell Jay Hendel, and the proposer.