ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit all new problem proposals and their solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468, or by email at reuler@nwmissouri.edu. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by November 15, 2013. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2,\ \beta = (1-\sqrt{5})/2,\ F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1126 **Proposed by José Gibergans-Báguena and José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain**

Let n be a positive integer. Prove that

$$\frac{1}{n}\sqrt[n]{\prod_{i=1}^{n}\left(1+\frac{F_nF_{n+1}}{F_i^2}\right)} \geq 1 + \sqrt[n]{\prod_{i=1}^{n}\frac{F_i^2}{F_nF_{n+1}}}.$$

B-1127 Proposed by George A. Hisert, Berkeley, California

Prove that, for any integer $n > 2$,

$$-17F_{n-2}^4 + 57F_{n-1}^4 + 402F_n^4 + 113F_{n+1}^4 - 25F_{n+2}^4 = 2F_{n-3}^2 L_{n+3}^2 \tag{1}$$

and

$$-17L_{n-2}^4 + 57L_{n-1}^4 + 402L_n^4 + 113L_{n+1}^4 - 25L_{n+2}^4 = 50L_{n-3}^2 F_{n+3}^2. \tag{2}$$

B-1128 Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain

Let n be a positive integer. Prove that

$$\left(\frac{F_{n+2}^2}{1+F_{n+1}^2}\right)\left(\frac{1-F_n^{-2}-F_{n+1}^{-2}}{F_n^2}\right) < \frac{\sqrt{3}}{4}.$$

B-1129 Proposed by D. M. Bătineţu–Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania

Prove that

$$2(L_{n+2}-3)^2 \le (5F_{2n+1}-4)n \tag{1}$$

and

$$2(F_{n+2}-1)^2 \le nF_{2n+1} \tag{2}$$

for any positive integer n.

B-1130 Proposed by D. M. Bătineţu–Giurgiu, Neculai Stanciu, and Gabriel Tica, Romania

Prove that

$$\sum_{k=1}^{n} \frac{F_k^{2m+2}}{k^{3m}} \ge \frac{4^m F_n^{m+1} F_{n+1}^{m+1}}{n^{2m}(n+1)^{2m}}$$

for all positive real numbers m.

SOLUTIONS

It Adds Up To Naught

B-1106 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 50.2, May 2012)

Prove that

$$\sum_{k=1}^{3n} F_{2F_k} \equiv 0 \pmod 5.$$

Solution by Chanequa Roy, University of South Carolina

By induction on n.
Base case: Let $n = 1$. Then

$$\begin{aligned}\sum_{k=1}^{3} F_{2F_k} &= F_{2F_1} + F_{2F_2} + F_{2F_3} \\ &= F_1 + F_1 + F_4 \\ &= 1 + 1 + 3 \equiv 0 \pmod 5.\end{aligned}$$

Assume that for $n = m$,

$$\sum_{k=1}^{3m} F_{2F_k} \equiv 0 \pmod 5.$$

We want to show that

$$\sum_{k=1}^{3(m+1)} F_{2F_k} \equiv 0 \pmod 5.$$

By the induction assumption,

$$\begin{aligned}\sum_{k=1}^{3(m+1)} F_{2F_k} &= \sum_{k=1}^{3m} F_{2F_k} + [F_{2F_{3m+1}} + F_{2F_{3m+2}} + F_{2F_{3m+3}}] \\ &\equiv [F_{2F_{3m+1}} + F_{2F_{3m+2}} + F_{2F_{3m+3}}] \pmod 5.\end{aligned}$$

Using Freitag's Identity [1], $F_{2n} \equiv n(-1)^{n+1} \pmod 5$, we obtain

$$\begin{aligned}&[F_{2F_{3m+1}} + F_{2F_{3m+2}} + F_{2F_{3m+3}}] \\ &\quad \equiv [F_{3m+1}(-1)^{F_{3m+1}+1} + F_{3m+2}(-1)^{F_{3m+2}+1} + F_{3m+3}(-1)^{F_{3m+3}+1}] \pmod 5.\end{aligned}$$

The period of the Fibonacci sequence modulus 2 is three, i.e. $(1, 1, 0, 1, 1, 0, \ldots)$. We know that $3m+3$ is divisible by three, so F_{3m+3} is even and that F_{3m+1} and F_{3m+2} area odd. Thus,

$$\begin{aligned}&[F_{3m+1}(-1)^{F_{3m+1}+1} + F_{3m+2}(-1)^{F_{3m+2}+1} + F_{3m+3}(-1)^{F_{3m+3}+1}] \\ &\quad \equiv [F_{3m+1} + F_{3m+2} - F_{3m+3}] \pmod 5.\end{aligned}$$

Using the fact that $F_n = F_{n-1} + F_{n-2}$, we know that $(F_{3m+1}) + (F_{3m+2}) - (F_{3m+3}) = 0$. Therefore,

$$\sum_{k=1}^{3(m+1)} F_{2F_k} \equiv 0 \pmod 5.$$

References

[1] H. T. Freitag, *Problem B-379*, The Fibonacci Quarterly, **17.2** (1979), 186.

Also solved by Paul S. Bruckman, Dmitry Fleishman, Russell J. Hendel, Robinson Higuita, Harris Kwong, ONU-Solve Problem Group, David Stong and John Hawkins (jointly), and the proposer.

Evaluating a Double Sum with Inverse Fibonacci Numbers

B-1107 Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 50.2, May 2012)

Determine

$$\sum_{j\ge 1, k\ge 3} \frac{1}{F_k^{4j-2}}.$$

Solution by Robinson Higuita (student), Universidad de Antioquia, Columbia

Since $|\frac{1}{F_k}| < 1$ for $k > 2$, we have that $\sum_{j=1}^{\infty} (\frac{1}{F_k^4})^j = \frac{1}{F_{k-1}^4}$. This implies that

$$\sum_{j\ge 1, k\ge 3} \frac{1}{F_k^{4j-2}} = \sum_{k\ge 3} F_k^2 \sum_{j\ge 1} \left(\frac{1}{F_k^4}\right)^j = \sum_{k\ge 3} F_k^2 \left(\frac{1}{F_k^4 - 1}\right). \tag{1}$$

From [2] we know the Gelin-Cesàro Identity

$$F_n^4 - 1 = F_{n-2}F_{n-1}F_{n+1}F_{n+2}.$$

This and (1) imply that

$$\sum_{j\ge 1, k\ge 3} \frac{1}{F_k^{4j-2}} = \sum_{k\ge 3} \left(\frac{F_k^2}{F_{k-2}F_{k-1}F_{k+1}F_{k+2}}\right). \tag{2}$$

It is easy to see that $F_k^2 = F_{k-2}F_{k+2} - F_{k-2}F_{k+1} + F_{k-1}F_{k+2} - F_{k-1}F_{k+1}$. Thus,

$$\begin{aligned}
\frac{F_k^2}{F_{k-2}F_{k-1}F_{k+1}F_{k+2}} &= \frac{F_{k-2}F_{k+2} - F_{k-2}F_{k+1} + F_{k-1}F_{k+2} - F_{k-1}F_{k+1}}{F_{k-2}F_{k-1}F_{k+1}F_{k+2}} \\
&= \frac{\cancel{F_{k-2}}\cancel{F_{k+2}}}{\cancel{F_{k-2}}F_{k-1}F_{k+1}\cancel{F_{k+2}}} - \frac{\cancel{F_{k-2}}\cancel{F_{k+1}}}{\cancel{F_{k-2}}F_{k-1}\cancel{F_{k+1}}F_{k+2}} \\
&\quad + \frac{\cancel{F_{k-1}}\cancel{F_{k+2}}}{F_{k-2}\cancel{F_{k-1}}F_{k+1}\cancel{F_{k+2}}} - \frac{\cancel{F_{k-1}}\cancel{F_{k+1}}}{F_{k-2}\cancel{F_{k-1}}\cancel{F_{k+1}}F_{k+2}} \\
&= \frac{1}{F_{k-1}F_{k+1}} - \frac{1}{F_{k-1}F_{k+2}} + \frac{1}{F_{k-2}F_{k+1}} - \frac{1}{F_{k-2}F_{k+2}}.
\end{aligned}$$

This and (2) imply that

$$\sum_{j\geq 1,k\geq 3}\frac{1}{F_k^{4j-2}}=\sum_{k\geq 3}\frac{1}{F_{k-1}F_{k+1}}+\sum_{k\geq 3}\left(\frac{1}{F_{k-2}F_{k+1}}-\frac{1}{F_{k-1}F_{k+2}}\right)-\sum_{k\geq 3}\frac{1}{F_{k-2}F_{k+2}}. \quad (3)$$

From [1, p. 442, exercises 35 and 36] we know that

$$\sum_{k\geq 3}\frac{1}{F_{k-1}F_{k+1}}=\sum_{k\geq 1}\frac{1}{F_kF_{k+2}}-\frac{1}{2}=\frac{1}{2}$$

$$\sum_{k\geq 3}\frac{1}{F_{k-2}F_{k+2}}=\sum_{k\geq 1}\frac{1}{F_kF_{k+4}}=\frac{7}{18}.$$

Note that

$$\sum_{k\geq 3}\left(\frac{1}{F_{k-2}F_{k+1}}-\frac{1}{F_{k-1}F_{k+2}}\right)=\frac{1}{F_1F_4}=\frac{1}{3}$$

because it is a telescopy series.

These and (3) imply that

$$\sum_{j\geq 1,k\geq 3}\frac{1}{F_k^{4j-2}}=\frac{1}{2}+\frac{1}{3}-\frac{7}{18}=\frac{4}{9}.$$

References

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley, New York, 2001.
[2] E. W. Weisstein, Gelin-Cesàro, MathWorld a Wolfram Web Resource,
http://mathworld.wolfram.com/Gelin-CesaroIdentity.html.

Also solved by Paul S. Bruckman, Dmitry Fleischman, and the proposer.

A Fibonacci and Triangular Numbers Inequality

B-1108 Proposed by D. M. Bătineţu–Giurgiu, Matei Basarab National College, Bucharest and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania
(Vol. 50.2, May 2012)

Let $T_k=\frac{k(k+1)}{2}$ for all $k\geq 1$. Prove that

$$\sum_{k=1}^{n}\frac{F_k^{2m+2}}{T_k^m}\geq\frac{3^m(F_nF_{n+1})^{m+1}}{n^mT_{n+1}^m}$$

for any positive integer $n\geq 1$ and for any positive real number m.

Solution by Paul S. Bruckman, Nanaimo, BC, Canada

Let $S=S_n=\sum_{k=1}^{n}F_k^2$, and $U=U_n=\sum_{k=1}^{n}T_k$. It is well-known that $S=F_nF_{n+1}$, and it is easily derived that

$$U_n=\frac{n(n+1)(n+2)}{6}=\frac{nT_{n+1}}{3}.$$

We see that we may write the desired inequality as follows:

$$S^{m+1} \le U^m \sum_{k=1}^{n} F_k^2 \left(\frac{F_k^2}{T_k}\right)^m. \tag{1}$$

Now Hölder's Inequality for sums states that for any two nonnegative sequences $\{a_k\}_{k\ge1}$ and $\{b_k\}_{k\ge1}$, we have for all $n \ge 1$, and for all $p > 0$ and $q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$:

$$\sum_{k=1}^{n} a_k b_k \le \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} b_k^q\right)^{1/q}. \tag{2}$$

In particular, take $\frac{1}{p} = \frac{m}{m+1}$, $\frac{1}{q} = \frac{1}{m+1}$, where m is any positive number. Also, take $a_k = T_k^{1/p}$, $b_k = F_k^2/T_k^{1/p}$. Then Hölder's Inequality takes the form:

$$\sum_{k=1}^{n} F_k^2 \le \left(\sum_{k=1}^{n} T_k\right)^{m/(m+1)} \left(\sum_{k=1}^{n} \frac{F_k^{2m+2}}{T_k^m}\right)^{1/(m+1)},$$

or

$$S \le U^{m/(m+1)} \left(\sum_{k=1}^{n} F_k^2 \left(\frac{F_k^2}{T_k}\right)^m\right)^{1/(m+1)}.$$

Raising both sides to the power $q = m + 1$ yields the desired inequality in (1).

Also solved by Dmitry Fleischman, Russell J. Hendel, Robinson Higuita, (student) and the proposer.

Candido's Identity Inspired Inequalities

B-1109 Proposed by D. M. Bătineţu–Giurgiu, Matei Basarab National College, Bucharest and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania
(Vol. 50.2, May 2012)

Prove that

$$2(F_n^4 + F_{n+1}^4 + F_{n+2}^4) > 9(F_n F_{n+1} F_{n+2})^{\frac{4}{3}}; \tag{1}$$

$$(F_n^2 + F_{n+1}^2 + F_{n+2}^2)^2 > \sqrt{6}(F_n F_{n+1} F_{n+2})^{\frac{2}{3}}; \tag{2}$$

$$(F_n^2 + F_{n+1}^2 + F_{n+2}^2)^2 \left(\frac{1}{F_n^4} + \frac{1}{F_{n+1}^4} + \frac{1}{F_{n+2}^4}\right) > 18; \tag{3}$$

$$2(F_n^4 + F_{n+1}^4 + F_{n+2}^4) \left(\frac{1}{F_n^2} + \frac{1}{F_{n+1}^2} + \frac{1}{F_{n+2}^2}\right)^2 > 81. \tag{4}$$

Solution by Kenneth B. Davenport, Dallas, PA

To solve the inequalities we will rely on Candido's Identity [1]

$$(F_n^2 + F_{n+1}^2 + F_{n+2}^2)^2 = 2(F_n^4 + F_{n+1}^4 + F_{n+2}^4).$$

To prove (1) we replace the LHS with $(F_n^2 + F_{n+1}^2 + F_{n+2}^2)$ and then take the square root of both sides to get

$$(F_n^2 + F_{n+1}^2 + F_{n+2}^2) > 3(F_n F_{n+1} F_{n+2})^{2/3}.$$

This follows immediately from using the AM-GM inequality for three numbers

$$\frac{a_1 + a_2 + a_3}{3} \geq (a_1 a_2 a_3)^{1/3}$$

with $a_1 = F_n^2$; $a_2 = F_{n+1}^2$; and $a_3 = F_{n+2}^2$. This completes the proof of (1).

To prove (2), we can establish the relation by showing a much stronger result, namely that

$$(F_n^2 + F_{n+1}^2 + F_{n+2}^2)^2 > 6(F_n F_{n+1} F_{n+2})^{4/3}.$$

Using the AM-GM inequality, we have

$$\frac{F_n^4 + F_{n+1}^4 + F_{n+2}^4}{3} > (F_n^4 F_{n+1}^4 F_{n+2}^4)^{1/3}.$$

Multiplying both sides by 6 and using once again Candido's Identity, we get the stronger inequality.

In part (3) we use Candido's Identity and rewrite the inequality as

$$\frac{F_n^4 + F_{n+1}^4 + F_{n+2}^4}{3} \left(\frac{1}{F_n^4} + \frac{1}{F_{n+1}^4} \frac{1}{F_{n+2}^4} \right) > 3.$$

This result follows immediately from the application of the AM-HM inequality for three numbers,

$$\frac{a_1 + a_2 + a_3}{3} > \frac{3}{\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \right)}.$$

To prove (4), we restate the inequality, using Candido's Identity again, as

$$(F_n^2 + F_{n+1}^2 + F_{n+2})^2 \left(\frac{1}{F_n^2} + \frac{1}{F_{n+1}^2} + \frac{1}{F_{n+2}^2} \right)^2 > 81.$$

Taking the square root of both sides, we obtain

$$(F_n^2 + F_{n+1}^2 + F_{n+2}^2) \left(\frac{1}{F_n^2} + \frac{1}{F_{n+1}^2} + \frac{1}{F_{n+2}^2} \right) > 9.$$

Again as in (3), this follows from the AM-HM inequality.

References

[1] R. S. Melham, *Ye olde Fibonacci curiosity shoppe revisited*, The Fibonacci Quarterly, **42.2** (2004), p. 155.

Also solved by Paul S. Bruckman, Dmitry Fleischman, Amos E. Gera, Russell J. Hendel, Zbigniew Jakubczyk, and the proposer.

Closed Forms for Sums of Squares

B-1110 Proposed by Sergio Falcón and Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain (Vol. 50.2, May 2012)

For any positive integer k, the k-Fibonacci and k-Lucas sequences, $\{F_{k,n}\}_{n\in\mathbb{N}}$ and $\{L_{k,n}\}_{n\in\mathbb{N}}$, both are defined recursively by $u_{n+1} = ku_n + u_{n-1}$ for $n \geq 1$, with respective initial conditions $F_{k,0} = 0$; $F_{k,1} = 1$ and $L_{k,0} = 2$; $L_{k,1} = k$. Prove that

$$\sum_{i\geq 0} \binom{2n}{i} F_{k,i}^2 = (k^2+4)^{n-1} L_{k,2n} \tag{1}$$

$$\sum_{i\geq 0} \binom{2n+1}{i} F_{k,i}^2 = (k^2+4)^{n} F_{k,2n+1} \tag{2}$$

$$\sum_{i\geq 0} \binom{2n}{i} L_{k,i}^2 = (k^2+4)^{n} L_{k,2n} \tag{3}$$

$$\sum_{i\geq 0} \binom{2n+1}{i} L_{k,i}^2 = (k^2+4)^{n+1} F_{k,2n+1}. \tag{4}$$

Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY and Ralph P. Grimaldi, Rose-Hulman Institute of Technology, Terre Haute, Indiana (separately)

It is a routine exercise to derive the Binet's formulas

$$F_{k,n} = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_{k,n} = \alpha^n + \beta^n,$$

where $\alpha = (k+\sqrt{k^2+4})/2$ and $\beta = (k-\sqrt{k^2+4})/2$. Since $\alpha\beta = -1$, we find $1+\alpha^2 = \alpha(\alpha-\beta)$, and $1+\beta^2 = -\beta(\alpha-\beta)$. Therefore, for any integer $m \geq 0$,

$$\begin{aligned}\sum_{i\geq 0} \binom{m}{i} (\alpha^i \pm \beta^i)^2 &= \sum_{i\geq 0} \binom{m}{i} [\alpha^{2i} \pm 2(\alpha\beta)^i + \beta^{2i}] \\ &= (1+\alpha^2)^m \pm 2(1+\alpha\beta)^m + (1+\beta^2)^m \\ &= (\alpha-\beta)^m[\alpha^m + (-\beta)^m] \\ &= \begin{cases} (\alpha-\beta)^m L_{k,m} & \text{if } m \text{ is even,} \\ (\alpha-\beta)^{m+1} F_{k,m} & \text{if } m \text{ is odd.} \end{cases}\end{aligned}$$

This leads to

$$\begin{aligned}(\alpha-\beta)^2 \sum_{i\geq 0} \binom{2n}{i} F_{k,i}^2 &= \sum_{i\geq 0} \binom{2n}{i} (\alpha^i - \beta^i)^2 = (\alpha-\beta)^{2n} L_{k,2n}, \\ (\alpha-\beta)^2 \sum_{i\geq 0} \binom{2n+1}{i} F_{k,i}^2 &= \sum_{i\geq 0} \binom{2n+1}{i} (\alpha^i - \beta^i)^2 = (\alpha-\beta)^{2n+2} F_{k,2n+1}, \\ \sum_{i\geq 0} \binom{2n}{i} L_{k,i}^2 &= \sum_{i\geq 0} \binom{2n}{i} (\alpha^i + \beta^i)^2 = (\alpha-\beta)^{2n} L_{k,2n}, \\ \sum_{i\geq 0} \binom{2n+1}{i} L_{k,i}^2 &= \sum_{i\geq 0} \binom{2n+1}{i} (\alpha^i + \beta^i)^2 = (\alpha-\beta)^{2n+2} F_{k,2n+1}.\end{aligned}$$

The results follow immediately from $(\alpha - \beta^2) = k^2 + 4$.

Also solved by Paul S. Bruckman, Dmitry Fleischman, Amos E. Gera, Russell J. Hendel, Zbigniew Jakubczyk, and the proposer.

We would like to acknowledge belatedly the solutions to problems B-1101, B-1102, B-1103, B-1104 by Amos E. Gera.