# ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
HARRIS KWONG

*Please submit solutions and problem proposals to Dr. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY, 14063, or by email at kwong@fredonia.edu. If you wish to have receipt of your submission acknowledged by mail, please include a self-addressed, stamped envelope.*

*Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by November 15, 2018. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results."*

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## BASIC FORMULAS

The Fibonacci numbers $F_n$ and the Lucas numbers $L_n$ satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

## PROBLEMS PROPOSED IN THIS ISSUE

**B-1226 Proposed by Ángel Plaza and Sergio Falcón, Universidad de Las Palmas de Gran Canaria, Spain.**

For any positive integer $k$, the $k$-Fibonacci and $k$-Lucas sequences, denoted $\{F_{k.n}\}_{n\geq 0}$ and $\{L_{k,n}\}_{n\geq 0}$ respectively, are both defined recursively by $u_{n+1} = ku_n + u_{n-1}$ for $n \geq 1$, with initial conditions $F_{k,0} = 0$, $F_{k,1} = 1$, and $L_{k,0} = 2$, $L_{k,1} = k$. Prove that, for any positive integer $n$,

$$\sum_{j=1}^{n} F_{k,j}^2 F_{k,2j} = \frac{F_{k,n}^2 F_{k,n+1}^2}{k}.$$

**B-1227 Proposed by Kenny B. Davenport, Dallas, PA.**

Let $C_n = \frac{1}{n+1}\binom{2n}{n}$ denote the $n$th Catalan number. Find the closed form expressions for the sums

$$\sum_{n=0}^{\infty} \frac{C_n F_n}{8^n} \qquad \text{and} \qquad \sum_{n=0}^{\infty} \frac{C_n L_n}{8^n}.$$

**B-1228** **Proposed by Hideyuki Ohtsuka, Saitama, Japan.**

For any integer $n \geq 0$, find the closed form expressions for the sums

(i) $S_n = \sum_{i=0}^{n}\sum_{j=0}^{n} L_{3^i}L_{3^j}L_{2(3^i-3^j)}$;

(ii) $T_n = \sum_{i=0}^{n}\sum_{j=0}^{n} F_{2\cdot 5^i}F_{2\cdot 5^j}L_{3(5^i-5^j)}$.

**B-1229** **Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania, and Neculai Stanciu, George Emil Palade School, Bazău, Romania.**

Let $m, p \geq 0$. Evaluate

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{\big((2n+1)!!\big)^{m+1}F_{n+1}^{p(m+1)}}}{(n+1)^m} - \frac{\sqrt[n]{\big((2n-1)!!\big)^{m+1}F_{n}^{p(m+1)}}}{n^m}\right),$$

and

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{\big((2n+1)!!\big)^{m+1}L_{n+1}^{p(m+1)}}}{(n+1)^m} - \frac{\sqrt[n]{\big((2n-1)!!\big)^{m+1}L_{n}^{p(m+1)}}}{n^m}\right).$$

**B-1230** **Proposed by T. Goy, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.**

For all integers $n \geq 0$, prove that

$$F_{2n+1} = (-1)^n \sum_{\substack{t_1,t_2,\ldots,t_n\geq 0\\ t_1+2t_2+\cdots+nt_n=n}} (-1)^{t_1+t_3+\cdots+t_{n-[1+(-1)^n]/2}}\frac{(t_1+t_2+\cdots+t_n)!}{t_1!\,t_2!\,\cdots\,t_n!}\cdot 2^{t_1}.$$

## SOLUTIONS

### Two Doses of AM-GM Inequality

**B-1206** **Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain. (Vol. 55.2, May 2017)**

Let $n \geq 2$ be an integer. Prove that

$$1 + \frac{1}{n^2}\sum_{1\leq i<j\leq n}\frac{\left(\sqrt{F_iF_{j+1}} - \sqrt{F_{i+1}F_j}\,\right)^2}{F_iF_j} \leq \frac{1}{n}\sum_{k=1}^{n}\frac{F_{k+1}}{F_k},$$

in which the subscripts are taken modulo $n$.

**Solution by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.**

To begin, we rewrite the given inequality in the form

$$n^2+\sum_{1\le i<j\le n}\left(\frac{F_{i+1}}{F_i}+\frac{F_{j+1}}{F_j}-2\sqrt{\frac{F_{i+1}}{F_j}\cdot\frac{F_{j+1}}{F_j}}\right)\le n\cdot\sum_{k=1}^{n}\frac{F_{k+1}}{F_k}.$$

Next, we see that

$$\sum_{1\le i<j\le n}\left(\frac{F_{i+1}}{F_i}+\frac{F_{j+1}}{F_j}\right)=(n-1)\sum_{k=1}^{n}\frac{F_{k+1}}{F_k},$$

$$\sum_{1\le i<j\le n}\sqrt{\frac{F_{i+1}}{F_i}\cdot\frac{F_{j+1}}{F_j}}\ge\frac{n(n-1)}{2}\cdot\sqrt[\frac{n(n-1)}{2}]{\prod_{1\le i<j\le n}\sqrt{\frac{F_{i+1}}{F_i}\cdot\frac{F_{j+1}}{F_j}}}=\frac{n(n-1)}{2},$$

$$\sum_{k=1}^{n}\frac{F_{k+1}}{F_k}\ge n\cdot\sqrt[n]{\prod_{k=1}^{n}\frac{F_{k+1}}{F_k}}=n,$$

in which the subscripts are taken modulo $n$. From it, follows that

$$\begin{aligned}
n^2+\sum_{1\le i<j\le n}\left(\frac{F_{i+1}}{F_j}+\frac{F_{j+1}}{F_i}-2\sqrt{\frac{F_{i+1}}{F_i}\cdot\frac{F_{j+1}}{F_j}}\right) &\le n^2+(n-1)\sum_{k=1}^{n}\frac{F_{k+1}}{F_k}-2\cdot\frac{n(n-1)}{2}\\
&= n+(n-1)\sum_{k=1}^{n}\frac{F_{k+1}}{F_k}\\
&\le \sum_{k=1}^{n}\frac{F_{k+1}}{F_k}+(n-1)\sum_{k=1}^{n}\frac{F_{k+1}}{F_k}\\
&= n\cdot\sum_{k=1}^{n}\frac{F_{k+1}}{F_k}.
\end{aligned}$$

**Also solved by Brian Bradie, Dmitry Fleischman, Hideyuki Ohtsuka, Albert Stadler, and the proposer.**

## Bergström on a Cyclic Sum

**B-1207 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania, and Neculai Stanciu, George Emil Palade School, Buzău, Romania.**
**(Vol. 55.2, May 2017)**

Prove that

$$\frac{F_n^4+F_1^4}{F_n^2+F_1^2}+\sum_{k=1}^{n-1}\frac{F_k^4+F_{k+1}^4}{F_{2k+1}}>F_nF_{n+1}$$

for any integer $n>1$.

**Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA.**

Let $x$ and $y$ be positive real numbers. Then

$$\frac{x^2+y^2}{x+y}-\frac{1}{2}(x+y)=\frac{2(x^2+y^2)-(x+y)^2}{2(x+y)}=\frac{(x-y)^2}{2(x+y)}\ge 0,$$

so that

$$\frac{x^2+y^2}{x+y} \geq \frac{1}{2}(x+y),$$

with equality holding if and only if $x = y$. Using this inequality and the identity $F_{2k+1} = F_k^2 + F_{k+1}^2$, it follows that

$$\begin{aligned}
\frac{F_n^4+F_1^4}{F_n^2+F_1^2} + \sum_{k=1}^{n-1} \frac{F_k^4+F_{k+1}^4}{F_{2k+1}} &= \frac{F_n^4+F_1^4}{F_n^2+F_1^2} + \sum_{k=1}^{n-1} \frac{F_k^4+F_{k+1}^4}{F_k^2+F_{k+1}^2} \\
&\geq \frac{1}{2}(F_n^2+F_1^2) + \frac{1}{2}\sum_{k=1}^{n-1}\left(F_k^2+F_{k+1}^2\right) \\
&= \sum_{k=1}^{n} F_k^2 \\
&= F_nF_{n+1}.
\end{aligned}$$

Note that equality holds for $n = 2$ as $F_1 = F_2 = 1$, but the inequality is strict for $n > 2$.

**Solution 2 by Wai-Kai Lai and John Risher (student) (jointly), University of South Carolina Salkehatchie, Walterboro, SC.**

Since $F_{2k+1} = F_k^2 + F_{k+1}^2$, and $F_nF_{n+1} = \sum_{k=1}^n F_k^2$, the claimed inequality is equivalent to

$$\frac{F_n^4+F_1^4}{F_n^2+F_1^2} + \frac{F_1^4+F_2^4}{F_1^2+F_2^2} + \cdots + \frac{F_{n-1}^4+F_n^4}{F_{n-1}^2+F_n^2} \geq \sum_{k=1}^{n} F_k^2.$$

Applying Bergström's inequality, we have

$$\frac{F_n^4}{F_n^2+F_1^2} + \frac{F_1^4}{F_1^2+F_2^2} + \cdots + \frac{F_{n-1}^4}{F_{n-1}^2+F_n^2} \geq \frac{(F_1^2+\cdots+F_n^2)^2}{2(F_1^2+\cdots+F_n^2)} = \frac{1}{2}\sum_{k=1}^{n} F_k^2,$$

$$\frac{F_1^4}{F_n^2+F_1^2} + \frac{F_2^4}{F_1^2+F_2^2} + \cdots + \frac{F_n^4}{F_{n-1}^2+F_n^2} \geq \frac{(F_1^2+\cdots+F_n^2)^2}{2(F_1^2+\cdots+F_n^2)} = \frac{1}{2}\sum_{k=1}^{n} F_k^2.$$

Summing these two inequalities we then have the desired claim. The equality holds only when $n = 2$.

**Also solved by Kenny B. Davenport, I. V. Fedak, Dmitry Fleischman, Donghae Lee (middle school student), Hideyuki Ohtsuka, Ángel Plaza, Henry Ricardo, Zachery R. Smith (student), and the proposer.**

## <u>Row Reduction on an Augmented Matrix</u>

**<u>B-1208</u> Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.**
**(Vol. 55.2, May 2017)**

For every positive integer $n$, find all real solutions of the following linear system of equations:

$$\begin{array}{cccccccccccl}
F_1x_1 & + & x_2 & & & & & & & & = & F_3, \\
F_2x_1 & + & F_1x_2 & + & x_3 & & & & & & = & F_4, \\
F_3x_1 & + & F_2x_2 & + & F_1x_3 & + & \cdots & & & & = & F_5, \\
\vdots & & \vdots & & \vdots & & \ddots & & & & & \vdots \\
F_{n-1}x_1 & + & F_{n-2}x_2 & + & F_{n-3}x_3 & + & \cdots & + & x_n & & = & F_{n+1}, \\
F_nx_1 & + & F_{n-1}x_2 & + & F_{n-2}x_3 & + & \cdots & + & F_1x_n & + \quad x_{n+1} & = & F_{n+2}, \\
F_{n+1}x_1 & + & F_nx_2 & + & F_{n-1}x_3 & + & \cdots & + & F_2x_n & + \quad F_1x_{n+1} & = & F_{n+3}-1.
\end{array}$$

**Composite solution by the proposer and the Elementary Problems Editor.**

We can use Gauss-Jordan elimination to solve the linear system. We want to apply row reduction to the following augmented matrix:

$$\left[\begin{array}{cccccc|c}
F_1 & 1 & 0 & \cdots & 0 & 0 & F_3 \\
F_2 & F_1 & 1 & \cdots & 0 & 0 & F_4 \\
F_3 & F_2 & F_1 & \cdots & 0 & 0 & F_5 \\
\vdots & \vdots & \vdots & \cdots & & & \vdots \\
F_{n-1} & F_{n-2} & F_{n-3} & \cdots & 1 & 0 & F_{n+1} \\
F_n & F_{n-1} & F_{n-2} & \cdots & F_1 & 1 & F_{n+2} \\
F_{n+1} & F_n & F_{n-1} & \cdots & F_2 & F_1 & F_{n+3}-1
\end{array}\right].$$

For $k = 3, 4, \ldots, n+1$, subtracting the sum of the first $k-2$ rows from row $k$ reduces the augmented matrix to (recall that $F_1 + F_2 + \cdots + F_m = F_{m+2} - 1$)

$$\left[\begin{array}{ccccccccc|c}
1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 2 \\
1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 3 \\
1 & 0 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 3 \\
1 & 0 & 0 & 1 & 1 & \cdots & 0 & 0 & 0 & 3 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots \\
1 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 3 \\
1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 2
\end{array}\right].$$

For $k = n, n-1, \ldots, 2, 1$, subtracting row $k$ from row $k+1$ further reduces the augmented matrix to

$$\left[\begin{array}{ccccccccc|c}
1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 2 \\
0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & -1
\end{array}\right].$$

We deduce that $x_2 = 2 - x_1$, $x_3 = x_n = 1$, and $x_k = x_{k+2}$ for $k = 2, 3, \ldots, n-1$. Therefore, if $n$ is even, the solution is

$$x_1 = x_2 = \cdots = x_{n+1} = 1;$$

but if $n$ is odd, the solution is

$$x_1 = t, \quad x_2 = x_4 = \cdots = x_{n+1} = 2 - t, \quad x_3 = x_5 = \cdots = x_n = 1,$$

where $t$ is an arbitrary real number.

**Also solved by Hsin-Yun Ching (student), Dmitry Fleischman, and Jason L. Smith.**

## A Bijective Argument on a Tribonacci Sum

**B-1209 Proposed by Hideyuki Ohtsuka, Saitama, Japan.**
**(Vol. 55.2, May 2017)**

The Tribonacci numbers $T_n$ satisfy $T_0 = 0$, $T_1 = T_2 = 1$, and

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \quad \text{for } n \geq 3.$$

Prove that

$$\sum_{k=1}^{n} T_{2k}T_{2k-1} = \left(\sum_{k=1}^{n} T_{2k-1}\right)^2$$

for any integer $n \geq 1$.

**Editor's Remark.** Most solvers first established the identity $T_{2n} + T_{2n-1} = 2\sum_{k=1}^{n} T_{2k-1}$, usually by induction, and used it to complete the proof. For instance, one may use induction again, and the inductive step may proceed as follows:

$$\begin{aligned}
\sum_{k=1}^{n+1} T_{2k}T_{2k-1} &= \left(\sum_{k=1}^{n} T_{2k}T_{2k-1}\right) + T_{2n+2}T_{2n+1} \\
&= \left(\sum_{k=1}^{n} T_{2k-1}\right)^2 + (T_{2n+1} + T_{2n} + T_{2n-1})T_{2n+1} \\
&= \left(\sum_{k=1}^{n} T_{2k-1}\right)^2 + T_{2n+1}^2 + 2\left(\sum_{k=1}^{n} T_{2k-1}\right)T_{2n+1} \\
&= \left[\left(\sum_{k=1}^{n} T_{2k-1}\right) + T_{2n+1}\right]^2 \\
&= \left(\sum_{k=1}^{n+1} T_{2k-1}\right)^2.
\end{aligned}$$

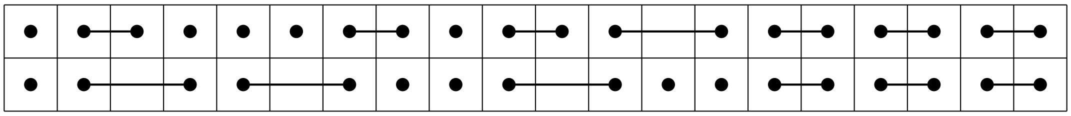
One solver submitted a bijective proof, using the polyominoes approach popularized by Arthur Benjamin and Jennifer Quinn.

**Solution by Charles Burnette, Academia Sinica, Taipei, Taiwan.**

Let $t_n$ denote the number of ways to tile a $1 \times n$ rectangular board with monominoes, dominoes, and trominoes, such that the leftmost piece placed on the board is a monomino. It is easy to see that $t_0 = 0$, $t_1 = 1$, $t_2 = 1$, and $t_n = t_{n-1} + t_{n-2} + t_{n-3}$ for $n \geq 3$; so that $t_n = T_n$ for all $n$.

We now count the number of ways to individually tile two $1 \times 2n$ rectangular boards, one atop the other, with (horizontal) monominoes, dominoes, and trominoes, such that (i) the leftmost piece placed on each board is a monomino, (ii) both tilings end on the right side with an identical (possibly empty) sequence of dominoes, and (iii) the immediate cell to the left of this sequence of dominoes on the bottom board is covered by a monomino. We shall call such a tiling *proper*.

Below is an example of a proper tiling with $n = 10$ (black dots connected by line segments mark the length of a tile).



First, we consider the location of the last monomino on the bottom board. Since both boards are of even length and end in sequences of dominoes, the bottom board's last monomino must occupy an even-numbered cell. If this monomino occupies cell $2k$, then there are $T_{2k}$ and $T_{2k-1}$ ways to properly tile the top and bottom boards, respectively. There are thus $\sum_{k=1}^{n} T_{2k}T_{2k-1}$ proper tilings altogether.

For an alternative enumeration, note that because the properly tiled boards are of even length and begin with a monomino, the top board must have at least one other odd-length tile piece.

If the last odd-length tile piece on the top board is a monomino occupying cell $2i$, then, to have a proper tiling, the last monomino on the bottom board can occupy cell $2j$ with $i \leq j \leq n$. In this case, the number of proper tilings satisfying this condition is $\sum_{1\leq i\leq j\leq n} T_{2i-1}T_{2j-1}$.

On the other hand, if the last odd-length tile piece on the top board is a tromino occupying cells $2i$ through $2i+2$, then, to again have a proper tiling, the last monomino on the bottom board can occupy cell $2j$ with $i+1 \leq j \leq n$. Hence, the number of proper tilings satisfying this condition is $\sum_{1\leq i<j\leq n} T_{2i-1}T_{2j-1}$.

Combining each case yields that

$$\sum_{k=1}^{n} T_{2k}T_{2k-1} = \sum_{1\leq i\leq j\leq n} T_{2i-1}T_{2j-1} + \sum_{1\leq i<j\leq n} T_{2i-1}T_{2j-1} = \left(\sum_{k=1}^{n} T_{2k-1}\right)^2.$$

**Also solved by Brian D. Beasley, Brian Bradie, Kenny B. Davenport, Steve Edwards, I. V. Fedak, Dmitry Fleischman, Robert Frontczak, Kantaphon Kuhapatanakul, Wei-Kai Lai and John Risher (student) (jointly), Carlos Montoya (student), Ángel Plaza, Raphael Schumacher (student), Albert Stadler, and the proposer.**

## Faà di Bruno to the Rescue!

**B-1210 Proposed by Taras Goy, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.**
**(Vol. 55.2, May 2017)**

Prove that

$$\sum_{t_1+2t_2+\cdots+nt_n=n} (-1)^{n-s} \frac{s!}{t_1!\,t_2!\,\cdots\,t_n!} F_1^{t_1} F_2^{t_2} \cdots F_n^{t_n} = \frac{1-(-1)^n}{2},$$

where $s = t_1 + t_2 + \cdots + t_n$.

**Solution 1 by Kantaphon Kuhapatanakul, Kasetsart University, Bangkok, Thailand.**

Merca [1] showed the formula for the determinant of the following Toeplitz-Hessenberg matrix as

$$\begin{vmatrix} a_1 & a_0 & 0 & \cdots & 0 & 0 \\ a_2 & a_1 & a_0 & \cdots & 0 & 0 \\ a_3 & a_2 & a_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & a_0 \\ a_n & a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{vmatrix} = \sum_{\substack{t_1+2t_2+\cdots+nt_n=n \\ t_1+t_2+\cdots+t_n=s}} (-a_0)^{n-s} \frac{s!}{t_1!\,t_2!\,\cdots\,t_n!} a_1^{t_1} a_2^{t_2} \cdots a_n^{t_n}.$$

Set $a_0 = 1$, and $a_i = F_i$ for $1 \leq i \leq n$, and using Problem B-1192 (Volume 54.3), we obtain the result.

**Solution 2 by Albert Stadler, Herrliberg, Switzerland.**

We assume that $n \geq 1$. Let

$$f(x) = \frac{1}{1+x}, \qquad \text{and} \qquad g(x) = \sum_{k=1}^{\infty} F_k x^k = \frac{x}{1-x-x^2}, \quad |x| < \frac{\sqrt{5}-1}{2}.$$

Then by the formula of Faà di Bruno for the $n$th derivative of a composite function, we find

$$\frac{1}{n!} D^n(f \circ g) = \sum_{t_1+2t_2+\cdots+nt_n=n} \frac{1}{t_1!\,t_2!\,\cdots\,t_n!} \cdot \left((D^{t_1+t_2+\cdots+t_n} f) \circ g\right) \cdot \prod_{m=1}^{n} \left(\frac{D^m g}{m!}\right)^{t_m}.$$

We have

$$\begin{aligned} D^{t_1+t_2+\cdots+t_n} f\Big|_{x=0} &= \frac{(-1)^{t_1+t_2+\cdots+t_n}(t_1+t_2+\cdots+t_n)!}{(1+x)^{t_1+t_2+\cdots+t_n+1}}\Bigg|_{x=0} \\ &= (-1)^{t_1+t_2+\cdots+t_n}(t_1+t_2+\cdots+t_n)!, \end{aligned}$$

and $\frac{D^m g}{m!}\Big|_{x=0} = F_m$. Therefore,

$$\frac{1}{n!} D^n(f \circ g)\Big|_{x=0} = \sum_{t_1+2t_2+\cdots+nt_n=n} \frac{(-1)^{t_1+t_2+\cdots+t_n}(t_1+t_2+\cdots+t_n)!}{t_1!\,t_2!\,\cdots\,t_n!} \prod_{m=1}^{n} F_m^{t_m}. \tag{1}$$

On the other hand,

$$(f \circ g)(x) = \frac{1}{1+\frac{x}{1-x-x^2}} = 1 - \frac{x}{1-x^2} = 1 + \frac{1}{2(1+x)} - \frac{1}{2(1-x)}$$

implies that

$$\frac{1}{n!} D^n(f \circ g)(x) = \frac{1}{n!}\left(\frac{(-1)^n\,n!}{2(1+x)^{n+1}} - \frac{n!}{2(1-x)^{n+1}}\right).$$

Hence,

$$\frac{1}{n!} D^n(f \circ g)\Big|_{x=0} = \frac{(-1)^n - 1}{2}. \tag{2}$$

The conclusion follows by comparing (1) and (2).

## References

[1] M. Merca, *A note on the determinant of Toeplitz-Hessenberg matrix*, Special Matrices. **1** (2013), 10–16.

**Also solved by I. V. Fedak, and the proposer.**