ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit solutions and problem proposals to Dr. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY, 14063, or by email at kwong@fredonia.edu. If you wish to have receipt of your submission acknowledged by mail, please include a self-addressed, stamped envelope.

Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by November 15, 2020. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results."

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

**<u>B-416</u> Proposed by Gene Jakubowski and V. E. Hoggatt, Jr., San Jose State University, San Jose, CA.
(Vol. 17.4, December 1979)**

Let F_n be defined for all integers (positive, negative, and zero) by $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$, and hence

$$F_n = F_{n+2} - F_{n+1}.$$

Prove that every positive integer m has at least one representation of the form

$$m = \sum_{j=-N}^{N} \alpha_j F_j,$$

with each α_j in $\{0, 1\}$ and $\alpha_j = 0$ when j is an integral multiple of 3.

Editor's Note: This is an old problem from more than 40 years ago. No solutions were received at the time it appeared, so we present the problem again, and invite the readers to solve it.

B-1266 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

For any positive integer n, prove that

$$\frac{F_{2n}}{F_{2n-1}} \geq \sqrt{1+\sqrt{1+\sqrt{1+\cdots+\sqrt{1+\sqrt{1}}}}} \qquad (n \text{ square roots}).$$

B-1267 Proposed by Robert Frontczak, Landesbank Baden-Württemberg, Stuttgart, Germany.

Prove that

$$\sum_{n=1}^{\infty} \frac{\zeta(2n)F_{2n}}{5^n} = -1 + \sum_{n=1}^{\infty} \frac{\zeta(2n)L_{2n}}{5^n} = \frac{\pi}{2\sqrt{5}} \tan\left(\frac{\pi}{2\sqrt{5}}\right),$$

where $\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}$ $(s > 1)$ is the Riemann zeta function.

B-1268 Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

Prove that, for $n \geq 1$,

(i) $L_{2n-1} = L_{2n-3} + 2L_{2n-5} + \cdots + (n-1)L_1 + 2n - 1$

(ii) $L_{2n} = L_{2n-2} + 2L_{2n-4} + \cdots + (n-1)L_2 + n + 2$

B-1269 Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

For all integers n and real numbers $x \leq y$, prove that

$$L_{n-1}(xF_n + yF_{n+2}) \leq xF_{n-2}F_{n+2} + 4yF_n^2.$$

B-1270 Proposed by Pridon Davlianidze, Tbilisi, Republic of Georgia.

Evaluate the following infinite products:

(A) $\displaystyle\prod_{n=2}^{\infty}\left(1 - \frac{5}{L_{2n-1}^2}\right)$

(B) $\displaystyle\prod_{n=2}^{\infty}\left(1 + \frac{5}{L_{2n}^2}\right)$

(C) $\displaystyle\prod_{n=2}^{\infty}\left(1 + \frac{5}{L_{2n-1}^2}\right)\left(1 - \frac{5}{L_{2n}^2}\right)$

(D) $\displaystyle\prod_{n=2}^{\infty}\left(1 - \frac{25}{L_n^4}\right)$

SOLUTIONS

Rationalize the Denominators

B-1246 Proposed by Raphael Schumacher (student), ETH Zurich, Switzerland. (Vol. 57.2, May 2019)

Prove that, for all integers $n \geq 0$,

$$\frac{F_{n-1}}{\sqrt{n+2}+\sqrt{n+1}} + \sum_{k=0}^{n} \frac{F_k}{\sqrt{k+1}+\sqrt{k}} = 2\sum_{k=1}^{n} \frac{F_{k-1}}{\sqrt{k+2}+\sqrt{k}} + \sqrt{2}-1,$$

and deduce that

$$\frac{F_{n+1}}{\sqrt{n+2}+\sqrt{n+1}} = \sum_{k=0}^{n}\left(\frac{2}{\sqrt{k+3}+\sqrt{k+1}} - \frac{1}{\sqrt{k+1}+\sqrt{k}}\right)F_k - \frac{F_n}{\sqrt{n+3}+\sqrt{n+2}} + \sqrt{2}-1.$$

Solution by Hideyuki Ohtsuka, Saitama, Japan.

We find

$$\begin{aligned}
&2\sum_{k=1}^{n} \frac{F_{k-1}}{\sqrt{k+2}+\sqrt{k}} - \sum_{k=0}^{n} \frac{F_k}{\sqrt{k+1}+\sqrt{k}} \\
&= \sum_{k=1}^{n}\left(\sqrt{k+2}-\sqrt{k}\right)F_{k-1} - \sum_{k=1}^{n}\left(\sqrt{k+1}-\sqrt{k}\right)F_k \\
&= \sum_{k=1}^{n}\left(\sqrt{k+2}\,F_{k-1} - \sqrt{k+1}\,F_{k-2}\right) - \sum_{k=1}^{n}\left(\sqrt{k+1}\,F_{k-1} - \sqrt{k}\,F_{k-2}\right) \\
&= \left(\sqrt{n+2}\,F_{n-1} - \sqrt{2}\,F_{-1}\right) - \left(\sqrt{n+1}\,F_{n-1} - F_{-1}\right) \\
&= \frac{F_{n-1}}{\sqrt{n+2}-\sqrt{n+1}} - \left(\sqrt{2}-1\right).
\end{aligned}$$

This proves the first identity, using which we deduce that

$$\begin{aligned}
&\sum_{k=0}^{n}\left(\frac{2}{\sqrt{k+3}-\sqrt{k+1}} - \frac{1}{\sqrt{k+1}+\sqrt{k}}\right)F_k + \sqrt{2}-1 \\
&= 2\sum_{k=1}^{n+1} \frac{F_{k-1}}{\sqrt{k+2}+\sqrt{k}} - \sum_{k=0}^{n} \frac{F_k}{\sqrt{k+1}+\sqrt{k}} + \sqrt{2}-1 \\
&= \frac{2F_n}{\sqrt{n+3}+\sqrt{n+1}} + \frac{F_{n-1}}{\sqrt{n+2}+\sqrt{n+1}} \\
&= \left(\sqrt{n+3}-\sqrt{n+1}\right)F_n + \left(\sqrt{n+2}-\sqrt{n+1}\right)F_{n-1} \\
&= \left(\sqrt{n+2}-\sqrt{n+1}\right)F_{n+1} + \left(\sqrt{n+3}-\sqrt{n+2}\right)F_n \\
&= \frac{F_{n+1}}{\sqrt{n+2}+\sqrt{n+1}} + \frac{F_n}{\sqrt{n+3}+\sqrt{n+2}}.
\end{aligned}$$

This proves the second identity.

Editor's Note: Frontczak generalized the result for any arbitrary arithmetic progression $\{a_n\}_{n\geq 1}$, where $a_n = a_1 + (n-1)d$. For example, he showed that

$$\frac{F_{n-1}}{\sqrt{a_{n+2}}+\sqrt{a_{n+1}}} + \sum_{k=0}^{n} \frac{F_k}{\sqrt{a_{k+1}}+\sqrt{a_k}} = 2\sum_{k=1}^{n} \frac{F_{k-1}}{\sqrt{a_{k+2}}+\sqrt{a_k}} + \frac{\sqrt{a_2}-\sqrt{a_1}}{d}.$$

Also solved by Michel Bataille, Brian Bradie, I. V. Fedak, Dmitry Fleischman, Robert Frontczak, Wei-Kai Lai, Ángel Plaza, Jason L. Smith, Albert Stadler, Santiago Alzate Suárez (student), Daniel Văcaru, and the proposer.

Sum of Products of Cubes of Consecutive Lucas Numbers

B-1247 Proposed by Kenny B. Davenport, Dallas, PA. (Vol. 57.2, May 2019)

Prove that, for all positive integers n,

$$\sum_{k=1}^{n} L_k^3 L_{k+1}^3 = \left(\sum_{k=1}^{n} L_k^2 L_{k+1}\right)^2 + 6\sum_{k=1}^{n} L_k^2 L_{k+1}.$$

Solution by Hideyuki Ohtsuka, Saitama, Japan.

We will prove the generalized identity

$$\sum_{k=1}^{n} G_k^3 G_{k+1}^3 = \left(\sum_{k=1}^{n} G_k^2 G_{k+1}\right)^2 + G_0G_1G_2\sum_{k=1}^{n} G_k^2 G_{k+1},$$

where the sequence $\{G_n\}$ is defined by $G_{n+2} = G_{n+1} + G_n$ for $n \geq 0$, with arbitrary G_0 and G_1. Since

$$G_kG_{k+1}G_{k+2} - G_{k-1}G_kG_{k+1} = G_kG_{k+1}(G_{k+2} - G_{k-1}) = G_kG_{k+1}\cdot 2G_k = 2G_k^2G_{k+1},$$

we have

$$\sum_{k=1}^{n} G_k^2 G_{k+1} = \frac{1}{2}\sum_{k=1}^{n}(G_kG_{k+1}G_{k+2} - G_{k-1}G_kG_{k+1}) = \frac{1}{2}\,G_nG_{n+1}G_{n+2} - \frac{1}{2}\,G_0G_1G_2.$$

We also have

$$\begin{aligned}(G_kG_{k+1}G_{k+2})^2 - (G_{k-1}G_kG_{k+1})^2 &= (G_kG_{k+1})^2\big(G_{k+2}^2 - G_{k-1}^2\big)\\ &= (G_kG_{k+1})^2(G_{k+2} - G_{k-1})(G_{k+2} + G_{k-1})\\ &= (G_kG_{k+1})^2\cdot 2G_k\cdot 2G_{k+1}\\ &= 4G_k^3G_{k+1}^3.\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{k=1}^{n} G_k^3 G_{k+1}^3 &= \sum_{k=1}^{n} \left[\left(\frac{1}{2}\, G_k G_{k+1} G_{k+2}\right)^2 - \left(\frac{1}{2}\, G_{k-1} G_k G_{k+1}\right)^2\right] \\
&= \left(\frac{1}{2}\, G_n G_{n+1} G_{n+2}\right)^2 - \left(\frac{1}{2}\, G_0 G_1 G_2\right)^2 \\
&= \left(\sum_{k=1}^{n} G_k^2 G_{k+1} + \frac{1}{2}\, G_0 G_1 G_2\right)^2 - \left(\frac{1}{2}\, G_0 G_1 G_2\right)^2 \\
&= \left(\sum_{k=1}^{n} G_k^2 G_{k+1}\right)^2 + G_0 G_1 G_2 \sum_{k=1}^{n} G_k^2 G_{k+1}.
\end{aligned}$$

Editor's Note: This problem is a Lucas analog of Problem B-1136, Volum 51.4 (2013).

Also solved by Michel Bataille, Brian Bradie, I. V. Fedak, Dmitry Fleischman, Robert Frontczak, Wei-Kai Lai and John Risher (student) (jointly), Ehren Metcalfe (computer proof), Ángel Plaza, Raphael Schumacher (student), Albert Stadler, David Terr, Daniel Văcaru, and the proposer.

An Inequality Derived from the Trapezoidal Rule

B-1248 Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.
(Vol. 57.2, May 2019)

For all positive integers n and a, prove that

$$\sum_{k=0}^{n} L_k (L_{k+1}^a + L_{k+2}^a) \leq (L_{n+2} - 1)(L_{n+2}^a + 1).$$

Solution by Brian Bradie, Christopher Newport University, Newport News, VA.

Replacing L_k with $L_{k+2} - L_{k+1}$, noting $L_1 = 1$, and dividing by 2, we see that the desired inequality is equivalent to

$$\sum_{k=0}^{n} (L_{k+2} - L_{k+1})\, \frac{L_{k+1}^a + L_{k+2}^a}{2} \leq (L_{n+2} - L_1)\, \frac{L_{n+2}^a + L_1^a}{2}.$$

The summation on the left side of this inequality is a trapezoidal rule approximation of the value of

$$\int_{L_1}^{L_{n+2}} x^a \, dx$$

using the $n+1$ subintervals $[L_1, L_2], [L_2, L_3], [L_3, L_4], \ldots, [L_{n+1}, L_{n+2}]$, whereas the expression on the right side is a trapezoidal rule approximation to the value of the same integral using just one subinterval $[L_1, L_{n+2}]$. Because $f(x) = x^a$ is convex on $[L_1, L_{n+2}]$ for all positive integers n and a, the desired inequality follows immediately.

Moreover, because $f(x) = x^a$ is convex on $[L_1, L_{n+2}]$ for all positive integers n and all real numbers $a \geq 1$ or $a \leq 0$, the inequality holds for all real numbers $a \geq 1$ or $a \leq 0$, with equality for $a = 1$ and $a = 0$. For $0 < a < 1$, the function $f(x) = x^a$ is concave on $[L_1, L_{n+2}]$, so the opposite inequality holds; that is,

$$\sum_{k=0}^{n}(L_{k+2} - L_{k+1})\,\frac{L_{k+1}^a + L_{k+2}^a}{2} > (L_{n+2} - L_1)\,\frac{L_{n+2}^a + L_1^a}{2}.$$

Editor's Note: This problem is a Lucas analog of Problem B-1223, Volume 56.1 (2018).

Also solved by Dmitry Fleischman, Wei-Kai Lai, I. V. Fedak, and the proposer.

The Tails of Two Series

B-1249 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 57.2, May 2019)

For positive integers s and t, prove that

$$\sum_{n=s}^{\infty}\frac{(-1)^{tn}}{\alpha^{(2t-1)n}F_n} = \sum_{n=t}^{\infty}\frac{(-1)^{sn}}{\alpha^{(2s-1)n}F_n}.$$

Solution by I. V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

Let $q = \frac{\beta}{\alpha}$, so that $|q| < 1$. Using $\alpha\beta = -1$, we obtain

$$\sum_{n=s}^{\infty}\frac{(-1)^{tn}}{\alpha^{(2t-1)n}F_n} = \sqrt{5}\sum_{n=s}^{\infty}\left(\frac{\beta}{\alpha}\right)^{tn}\frac{\alpha^n}{\alpha^n - \beta^n} = \sqrt{5}\sum_{n=s}^{\infty}\frac{q^{tn}}{1-q^n} = \sqrt{5}\sum_{n=s}^{\infty}q^{tn}\sum_{k=0}^{\infty}q^{kn}.$$

Similarly,

$$\sum_{n=t}^{\infty}\frac{(-1)^{sn}}{\alpha^{(2s-1)n}F_n} = \sqrt{5}\sum_{n=t}^{\infty}q^{sn}\sum_{k=0}^{\infty}q^{kn}.$$

Because

$$\begin{aligned}\sum_{n=s}^{\infty}q^{tn}\sum_{k=0}^{\infty}q^{kn} &= \sum_{n=s}^{\infty}\sum_{k=0}^{\infty}q^{(t+k)n} = \sum_{\ell=0}^{\infty}\sum_{k=0}^{\infty}q^{(t+k)(s+\ell)}\\ &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty}q^{(s+\ell)(t+k)} = \sum_{n=t}^{\infty}\sum_{\ell=0}^{\infty}q^{(s+\ell)n} = \sum_{n=t}^{\infty}q^{sn}\sum_{\ell=0}^{\infty}q^{\ell n},\end{aligned}$$

we have that

$$\sum_{n=s}^{\infty}\frac{(-1)^{tn}}{\alpha^{(2t-1)n}F_n} = \sum_{n=t}^{\infty}\frac{(-1)^{sn}}{\alpha^{(2s-1)n}F_n}.$$

Also solved by Michel Bataille, Raphael Schumacher (student), and the proposer.

Make It Telescopic!

B-1250 Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain. (Vol. 57.2, May 2019)

Evaluate

$$\sum_{k=1}^{\infty} \tan^{-1} \frac{F_{k+1}}{F_k F_{k+2} + 1} \tan^{-1} \frac{1}{F_{k+1}}.$$

Solution by Jason L. Smith, Richland Community College, Decatur, IL.

Note that the first factor in the sum is equal to

$$\begin{aligned} \tan^{-1} \frac{F_{k+2} - F_k}{F_k F_{k+2} + 1} &= \tan^{-1} F_{k+2} - \tan^{-1} F_k \\ &= \left(\frac{\pi}{2} - \tan^{-1} \frac{1}{F_{k+2}}\right) - \left(\frac{\pi}{2} - \tan^{-1} \frac{1}{F_k}\right) \\ &= \tan^{-1} \frac{1}{F_k} - \tan^{-1} \frac{1}{F_{k+2}}. \end{aligned}$$

Our sum now becomes

$$\begin{aligned} &\sum_{k=1}^{\infty} \left(\tan^{-1} \frac{1}{F_k} - \tan^{-1} \frac{1}{F_{k+2}}\right) \tan^{-1} \frac{1}{F_{k+1}} \\ &= \sum_{k=1}^{\infty} \left(\tan^{-1} \frac{1}{F_k} \tan^{-1} \frac{1}{F_{k+1}} - \tan^{-1} \frac{1}{F_{k+1}} \tan^{-1} \frac{1}{F_{k+2}}\right). \end{aligned}$$

This is a telescoping sum in which only the first term survives. Therefore, the sum evaluates to

$$\tan^{-1} \frac{1}{F_1} \tan^{-1} \frac{1}{F_2} = \left(\tan^{-1} 1\right)^2 = \frac{\pi^2}{16}.$$

Editor's Note: Frontczak noted that the following arctangent product involving Fibonacci numbers

$$\sum_{k=0}^{\infty} \tan^{-1} \left(\frac{\sqrt{5}}{L_{2k+1}}\right) \tan^{-1} \left(\frac{1}{\sqrt{5}\, F_{2k+1}}\right),$$

also converges to the same sum $(\tan^{-1} 1)^2$, and the convergence is faster than the series given in the problem. The claim follows from

$$\tan^{-1} \left(\frac{1}{\alpha^{2k}}\right) - \tan^{-1} \left(\frac{1}{\alpha^{2k+2}}\right) = \tan^{-1} \left(\frac{1}{\sqrt{5}\, F_{2k+1}}\right),$$

and

$$\tan^{-1} \left(\frac{1}{\alpha^{2k}}\right) + \tan^{-1} \left(\frac{1}{\alpha^{2k+2}}\right) = \tan^{-1} \left(\frac{\sqrt{5}}{L_{2k+1}}\right).$$

Also solved by Michel Bataille, Brian Bradie, Alejandro Cardona Castrillón (student), I. V. Fedak, Dmitry Fleischman, Robert Frontczak, Hideyuki Ohtsuka, Raphael Schumacher (student), Albert Stadler, Daniel Văcaru, Dan Weiner, and the proposer.