ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit all new problem proposals and their solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468, or by email at reuler@nwmissouri.edu. All solutions to others’ proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by February 15, 2015. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting “well-known results”.

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers $F_n$ and the Lucas numbers $L_n$ satisfy
\[ F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1; \]
\[ L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1. \]
Also, $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE


Calculate each of the following:
\[
\begin{align*}
(i) \quad & \lim_{n \to \infty} \left( n+1 \sqrt{(n+1)!F_{n+1}} - n \sqrt{n!F_n} \right), \\
(ii) \quad & \lim_{n \to \infty} \left( n+1 \sqrt{(n+1)!L_{n+1}} - n \sqrt{n!L_n} \right), \\
(iii) \quad & \lim_{n \to \infty} \left( n+1 \sqrt{(2n+1)!!F_{n+1}} - n \sqrt{(2n-1)!!F_n} \right), \\
(iv) \quad & \lim_{n \to \infty} \left( n+1 \sqrt{(2n+1)!!L_{n+1}} - n \sqrt{(2n-1)!!L_n} \right).
\end{align*}
\]
For any positive integer number \( k \), the \( k \)-Fibonacci and \( k \)-Lucas sequences, \( \{F_{k,n}\}_{n \in \mathbb{N}} \) and \( \{L_{k,n}\}_{n \in \mathbb{N}} \), both are defined recurrently by \( u_{n+1} = ku_n + u_{n-1} \) for \( n \geq 1 \) with respective initial conditions \( F_{k,0} = 0 \), \( F_{k,1} = 1 \), and \( L_{k,0} = 2 \), \( L_{k,1} = k \). Find a closed form expression for

\[
\sum_{n=1}^{\infty} \frac{F_{k,2^n}}{1 + L_{k,2^n+1}}
\]
as a function of \( \alpha = \frac{k + \sqrt{k^2 + 4}}{2} \).

For any positive integer number \( k \), the \( k \)-Fibonacci and \( k \)-Lucas sequences, \( \{F_{k,n}\}_{n \in \mathbb{N}} \) and \( \{L_{k,n}\}_{n \in \mathbb{N}} \), both are defined recurrently by \( u_{n+1} = ku_n + u_{n-1} \) for \( n \geq 1 \) with respective initial conditions \( F_{k,0} = 0 \), \( F_{k,1} = 1 \), and \( L_{k,0} = 2 \), \( L_{k,1} = k \). Prove that

\[
\sum_{i=0}^{n} \left( \frac{2}{k} \right)^i L_{k,i} = k \left( \frac{2}{k} \right)^{n+1} F_{k,n+1}.
\]

Find a closed form expression for

\[
\sum_{i=0}^{n} L_i^2 L_{i+1}^2.
\]

Prove each of the following:

(i) \[
\sum_{k=1}^{\infty} \frac{L_{2^{k+1}}}{F_3^{2^k}} = \frac{5}{4},
\]

(ii) \[
\sum_{k=1}^{\infty} \frac{F_{2^{k-1}}^2}{L_{2^k}^2 - 1} = \frac{3}{20}.
\]
THE FIBONACCI QUARTERLY

SOLUTIONS

Tedious But Pretty Identities

B-1131 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 51.3, August 2013)

For integers $a$ and $b$, prove that
\[
(F^2_a + F^2_{a+1} + F^2_{a+2})(F_aF_b + F_{a+1}F_{b+1} + F_{a+2}F_{b+2})
= 2(F^3_a + F^3_{a+1}F_{b+1} + F^3_{a+2}F_{b+2});
\]
\[
(1)
\]
and
\[
(F^2_a + F^2_{a+1} + F^2_{a+2})(F^3_aF_b + F^3_{a+1}F_{b+1} + F^3_{a+2}F_{b+2})
= (F^2_b + F^2_{b+1} + F^2_{b+2})(F^3_a + F^3_{a+1}F_{b+1} + F^3_{a+2}F_{b+2}).
\]
\[
(2)
\]

Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

For convenience, let $x, y, z$ denote $F_a, F_{a+1}, F_{a+2}$, and $u, v, w$ denote $F_b, F_{b+1}, F_{b+2}$, respectively. Using the equalities $z = x + y$ and $w = u + v$ we have
\[
\text{LHS} = 2(x^2 + y^2 + xy)(xu + yv + (x + y)(u + v))
= 2(2ux^3 + vx^3 + 3ux^2y + 3uxy^2 + 3vxy^2 + uy^3 + 2vy^3)
= 2(ux^3 + vy^3 + ux^3 + vx^3 + 3ux^2y + 3vxy^2 + 3uxy^2 + 3vxy^2 + uy^3 + vy^3)
= 2(x^3u + y^3v + (x + y)^3(u + v)
= \text{RHS}.
\]

In order to prove (2) it is enough to prove that
\[
(x^2 + y^2 + xy)(xu^3 + yv^3 + (x + y)(u + v))^3 = (u^2 + v^2 + uv)(x^3u + y^3v + (x + y)^3(u + v)).
\]
The expansion of both terms yields the same answer as can be checked with the aid of a software package like Mathematica or MATLAB.

Also solved by Kenneth B. Davenport, Dmitry Fleischman, Esref Gurel and Mustafa Asci (jointly), Russell J. Hendel, and the proposer.

The AM-GM Inequality Paves the Way

B-1132 Proposed by D. M. Bătinețu–Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade General School, Buzău, Romania.
(Vol. 51.3, August 2013)

Prove that
\[
(i) \quad F^2_n + F^2_{n+1} + 5F^2_{n+2} > 4\sqrt{6}\sqrt{F_nF_{n+1}F_{n+2}},
(ii) \quad L^2_n + L^2_{n+1} + 5L^2_{n+2} > 4\sqrt{6}\sqrt{L_nL_{n+1}L_{n+2}},
\]
for any positive integer $n$.

**Solution by Robinson Higuita (student), Universidad de Antioquia, Columbia.**

(i) From

\[ F_{n+4}^2 = (F_{n+2} + F_{n+3})^2 = (2F_n + 3F_{n+1})^2 = ((2F_n)^2 + (3F_{n+1})^2) + 12F_nF_{n+1} \]

and the inequality of arithmetic and geometric means we have that

\[ F_{n+4}^2 \geq 12F_nF_{n+1} + 12F_nF_{n+1} = 24F_nF_{n+1}. \]

That is, \( 2\sqrt{6F_nF_{n+1}} \leq F_{n+4} \). Therefore, \( 4\sqrt{6\sqrt{F_nF_{n+1}}} \leq 4F_{n+2}^2 + 2F_{n+1}F_{n+2} \).

This and the inequality of arithmetic and geometric mean imply \( 2F_{n+1}F_{n+2} \leq F_n^2 + F_{n+1}^2 \). So, \( 4\sqrt{6\sqrt{F_nF_{n+1}}} \leq 4F_{n+2}^2 + 2F_{n+1}F_{n+2} < 5F_{n+2}^2 + F_n^2 + F_{n+1}^2 \).

(ii) The proof of

\[ 4\sqrt{6\sqrt{F_nF_{n+1}}} \leq 4F_{n+2}^2 + 2F_{n+1}F_{n+2} < 5F_{n+2}^2 + F_n^2 + F_{n+1}^2 \]

is similar to the previous proof. It is enough to replace \( F_n \) with \( L_n \).

Also solved by Kenneth B. Davenport, Dmitry Freischman, Esref Gurel and Mustafa Asci (jointly), Russell Jay Hendel, Zbigniew Jakubczyk, and the proposer.

**The Value of a Series of Reciprocal Fibonacci Numbers**

**B-1133** Proposed by Mohammed K. Azarian, University of Evansville, Indiana. (Vol. 51.3, August 2013)

Determine the value of the following infinite series

\[ S = \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 3} - \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{3 \cdot 8} - \frac{1}{5 \cdot 8} + \frac{1}{8 \cdot 13} + \frac{1}{8 \cdot 21} - \frac{1}{13 \cdot 21} + \frac{1}{21 \cdot 34} + \]

\[ -\frac{1}{21 \cdot 55} - \frac{1}{34 \cdot 55} + \frac{1}{55 \cdot 89} + \frac{1}{55 \cdot 144} - \frac{1}{89 \cdot 144} + \frac{1}{144 \cdot 233} + \frac{1}{144 \cdot 377} - \ldots. \]

Solution by John D. Watson, Jr. (student), The Citadel, The Military College of South Carolina.

We start by expressing the original problem in terms of a series

\[ S = \sum_{k=1}^{\infty} \frac{1}{F_{2k}F_{2k+1}} + \frac{1}{F_{2k}F_{2k+2}} - \frac{1}{F_{2k+1}F_{2k+2}}. \]
Simplifying:

\[ S = \sum_{k=1}^{\infty} \frac{F_{2k+2} + F_{2k+1} - F_{2k}}{F_{2k}F_{2k+1}F_{2k+2}} \]

\[ = \sum_{k=1}^{\infty} \frac{2F_{2k}}{F_{2k}F_{2k+1}F_{2k+2}} \]

\[ = \sum_{k=1}^{\infty} \frac{2}{F_{2k}F_{2k+2}} \]

\[ = 2 \sum_{k=1}^{\infty} \frac{2}{F_{2k} + F_{2k+1}} \]

\[ = 2 \sum_{i=2}^{\infty} \frac{(-1)^i}{F_iF_{i+1}}. \]

Let \( S_n \) be the \( n \)th partial sum of this series. Since \( S_n \) is a telescoping sum, we have that

\[ S_n = 2 \sum_{i=2}^{n} \frac{(-1)^i}{F_iF_{i+1}} = 2 \left( \frac{F_1}{F_2} - \frac{F_n}{F_{n+1}} \right). \]

Therefore,

\[ S = 2 \lim_{n \to \infty} \left( \frac{F_1}{F_2} - \frac{F_n}{F_{n+1}} \right) = 2 \left( 1 - \frac{1}{\alpha} \right) = \frac{-2\beta}{\alpha} = 3 - \sqrt{5}. \]

Also solved by Brian D. Beasley, Kenneth B. Davenport, Dmitry Fleischman, Russell Jay Hendel, Robinson Higuita and Bilson Castro (jointly), Ángel Plaza, and the proposer.

**Easier Than It Looks**

**B-1134** Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain and Francesc Gispert Sánchez, CFIS, Barcelona Tech, Barcelona, Spain. (Vol. 51.3, August 2013)

Let \( n \) be a positive integer. Prove that

\[ \frac{1}{F_nF_{n+1}} \left[ \left( \frac{F_1}{F_2} - \frac{F_n}{F_{n+1}} \right) \sum_{k=1}^{n} F_k^{2n} + \prod_{k=1}^{n} F_k^2 \right] \geq \left( \prod_{k=1}^{n} F_k^{(1-1/n)} \right)^2. \]

Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

By the AM-GM inequality, the LHS of the given inequality satisfies
LHS \geq \frac{1}{F_n F_{n+1}} \left( (n - 1) \prod_{k=1}^{n} F_{k}^2 + \prod_{k=1}^{n} F_{k}^2 \right)
= \frac{1}{F_n F_{n+1}} \left( (n - 1) \prod_{k=1}^{n} F_{k}^2 + \prod_{k=1}^{n} F_{k}^2 \right)
= \frac{1}{F_n F_{n+1}} \left( \prod_{k=1}^{n} F_{k}^2 \right)
= \frac{n \prod_{k=1}^{n} F_{k}^2}{\sqrt{\prod_{k=1}^{n} F_{k}^2}} \quad \text{(since } F_n F_{n+1} = \sum_{i=1}^{n} F_i^2 \text{)}
\geq \frac{\prod_{k=1}^{n} F_{k}^2}{\sqrt{\prod_{k=1}^{n} F_{k}^2}} \quad \text{(by the AM-GM inequality)}
= \text{RHS.}

Also solved by Dmitry Fleischman and the proposer.

A General Inequality Applied to Fibonacci and Lucas Numbers


Prove that

\[ \frac{F_{n+1}^2}{F_n F_{n+1} + F_{n+2}^2} + \frac{F_{n+2}^2}{F_{n+1} F_{n+2} + F_n^2} + \frac{F_n^2}{F_{n+2}(F_n F_{n+1} + F_{n+2}^2)} > \frac{3}{2F_n F_{n+1} F_{n+2}}; \]  
(1)

and

\[ \frac{L_{n+1}^2}{L_n L_{n+1} + L_{n+2}^2} + \frac{L_{n+2}^2}{L_{n+1} L_{n+2} + L_n^2} + \frac{L_n^2}{L_{n+2}(L_n L_{n+1} + L_{n+2}^2)} > \frac{3}{2L_n L_{n+1} L_{n+2}}; \]  
(2)

for any positive integer \( n \).

Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

The proposed inequalities are particular cases of the following more general inequality for positive numbers \( x, y, z \):

\[ \frac{x^2}{z^3(xz + y^2)} + \frac{y^2}{x^3(xy + z^2)} + \frac{z^2}{y^3(yz + x^2)} \geq \frac{3}{2xyz}. \]
which may be written as
\[
\frac{x^3 y}{z^2(xz + y^2)} + \frac{y^3 z}{x^2(xy + z^2)} + \frac{z^3 x}{y^2(yz + x^2)} \geq \frac{3}{2}
\]
\[
\frac{x^2}{y + z} + \frac{y^2}{x + z} + \frac{z^2}{x + y} \geq \frac{3}{2}
\]
By changing variables \(a = \frac{x}{z}\), \(b = \frac{z}{y}\), and \(c = \frac{y}{x}\), the last inequality reads
\[
\frac{a^2}{b + c} + \frac{b^2}{a + c} + \frac{c^2}{a + b} \geq \frac{3}{2}.
\] (3)

Then, by Chebyshev’s sum inequality, the LHS of (3) satisfies
\[
\text{LHS} \geq \frac{a + b + c}{3} \left( \frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \right)
\]
\[
\geq \sqrt[3]{abc} \cdot \frac{3}{2} \quad (\text{by the AM-GM and Nesbitt inequalities})
\]
\[
= \frac{3}{2},
\]
since \(abc = 1\).

Also solved by Kenneth B. Davenport, Dmitry Fleischman, Russell Jay Hendel, and the proposer.

We would like to acknowledge Dmitry Fleischman for solving B-1127.