ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
RUSS EULER AND JAWAD SADEK

Please submit all new problem proposals and their solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468, or by email at reuler@nwmissouri.edu. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by November 15, 2016. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1186 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Prove that

$$\sum_{n=0}^{\infty} \frac{(-1)^n L_{2^n}}{L_{2^{n+1}}+1} = 0.$$

B-1187 Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.

Let $n \geq 1$ be a positive integer. Find all real solutions of the following system of equations:

$$\begin{aligned} x^3 + L_n x + y &= F_n(1 + L_n) + F_n x^2 \\ F_n y^3 + F_{2n} y + z &= F_n(1 + F_{2n}) + F_n^2 y^2 \\ L_n z^3 + L_n^2 z + x &= F_n(1 + L_n^2) + F_{2n} z^2. \end{aligned}$$

B-1188 Proposed by Kenny B. Davenport, Dallas, PA.

Find a closed form expression for

$$5\sum_{k=0}^{n} F_{3^k}^3 - 4\sum_{k=0}^{n} F_{3^k}.$$

B-1189 Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

Find a closed form for

$$\sum_{k=0}^{2n} \binom{2n}{k} L_{2n-2k}.$$

B-1190 Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.

Let $n \geq 1$ be a positive integer. Compute

$$\frac{F_{n+2}}{F_nF_{n+1}}\left(\frac{F_n^n + F_{n+1}^n - F_{n+2}^n}{F_n^{n-1} + F_{n+1}^{n-1} + F_{n+2}^{n-1}}\right) + \frac{F_{n+3}}{F_{n+1}F_{n+2}}\left(\frac{F_{n+1}^n + F_{n+2}^n - F_n^n}{F_n^{n-1} + F_{n+1}^{n-1} + F_{n+2}^{n-1}}\right) + \frac{2F_n + F_{n+1}}{F_{n+2}F_n}\left(\frac{F_{n+2}^n + F_n^n - F_{n+1}^n}{F_n^{n-1} + F_{n+1}^{n-1} + F_{n+2}^{n-1}}\right).$$

SOLUTIONS

Series of Reciprocals

B-1166 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 53.2, May 2015)

Prove that

$$\sum_{k=1}^{\infty} \frac{F_{2^{k-1}}}{L_{2^k} - 2} = \frac{15 - \sqrt{5}}{10}.$$

Solution by Jaroslav Seibert, University of Pardubice, Czech Republic.

First we note that $L_{2n} - 2(-1)^n = 5F_n^2$ [1, p. 97], and therefore $L_{2^k} - 2(-1)^{2^{k-1}} = L_{2^k} - 2 = 5F_{2^{k-1}}^2$ for any positive integer $k > 1$. Thus,

$$\begin{aligned} S &= \sum_{k=1}^{\infty} \frac{F_{2^{k-1}}}{L_{2^k} - 2} = \frac{F_1}{L_2 - 2} + \sum_{k=2}^{\infty} \frac{F_{2^{k-1}}}{L_{2^k} - 2} \\ &= 1 + \sum_{k=2}^{\infty} \frac{F_{2^{k-1}}}{L_{2^k} - 2} = 1 + \sum_{k=2}^{\infty} \frac{F_{2^{k-1}}}{5F_{2^{k-1}}^2} \\ &= 1 + \frac{1}{5}\sum_{k=2}^{\infty} \frac{1}{F_{2^{k-1}}} = \frac{4}{5} + \frac{1}{5}\sum_{k=1}^{\infty} \frac{1}{F_{2^{k-1}}} \\ &= \frac{4}{5} + \frac{1}{5}\sum_{k=0}^{\infty} \frac{1}{F_{2^k}}. \end{aligned}$$

However, it is known [1, p. 437] that $\sum_{k=0}^{\infty} \frac{1}{F_{2^k}} = 4 - \alpha = 3 + \beta$. Then we have

$$S = \frac{4}{5} + \frac{1}{5}(4 - \alpha) = \frac{4}{5} + \frac{1}{5}\left(4 - \frac{1+\sqrt{5}}{2}\right) = \frac{15 - \sqrt{5}}{10}.$$

References

[1] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley and Sons, Inc., 2001.

Also solved by Brian Bradie, Charles K. Cook, Steve Edwards, Dmitry Fleischman, Russell Jay Hendel, Zbigniew Jakubczyk, Harris Kwong, ONU Problem Solving Group, Ángel Plaza and Francisco Perdomo (jointly), Dan Weiner, Nazmieh Yilmaz, and the proposer.

Polygons with Fibonacci Number Coordinates

B-1167 Proposed by Atara Shriki and Opher Liba, Oranim College of Education, Israel.
(Vol. 53.2, May 2015)

Find the area of the polygon with vertices

$$(F_1, F_2), (F_3, F_4), (F_5, F_6), \cdots, (F_{2n-1}, F_{2n}) \quad \text{for} \quad n \geq 3.$$

Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY.

The polygon is convex, its interior can be partitioned into $n-2$ triangles, with vertices (F_1, F_2), (F_{2k-1}, F_{2k}) and (F_{2k+1}, F_{2k+2}), where $2 \leq k \leq n-1$. The area Δ_k of the triangle with these three vertices can be computed by comparing the areas of the trapezoids under its three sides (and above the x-axis). Since $(F_1, F_2) = (1, 1)$, we find

$$\begin{aligned}2\Delta_k &= (F_{2k-1}-1)(F_{2k}+1)+(F_{2k+1}-F_{2k-1})(F_{2k}+F_{2k+2})-(F_{2k+1}-1)(F_{2k+2}+1)\\ &= F_{2k}F_{2k+1}-F_{2k-1}F_{2k+2}+F_{2k-1}+F_{2k+2}-F_{2k}-F_{2k+1}\\ &= \frac{L_{4k+1}-1}{5}-\frac{L_{4k+1}+4}{5}+F_{2k-1}\\ &= F_{2k-1}-1.\end{aligned}$$

Therefore,

$$2\sum_{k=2}^{n-1}\Delta_k=\left(\sum_{k=2}^{n-1}F_{2k-1}\right)-(n-2)=(F_{2n-2}-1)-(n-2);$$

and the area of the polygon is $\frac{1}{2}(F_{2n-2}-n+1)$.

Also solved by Jeremiah Bartz, Brian D. Beasley, Brian Bradie, Charles K. Cook, Ravi Kumar Davala, Steve Edwards, David L. Farnsworth, Russell Jay Hendel, Ángel Plaza and Francisco Perdomo (jointly), Jaroslav Seibert, Jason L. smith, David Stone and John Hawkins (jointly), and the proposer.

The Minimum Value of a Sum

B-1168 Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain. (Vol. 53.2, May 2015)

Let n be a nonnegative integer. Find the minimum value of

$$\frac{F_n}{2F_n+3L_n(1+F_n)}+\frac{F_nL_n}{2F_nL_n+3(F_n+L_n)}+\frac{L_n}{2L_n+3F_n(1+L_n)}.$$

Solution by Brian Bradie, Department of Mathematics, Christopher Newport University, VA.

We make use of the following parametrized Nesbitt's inequality [1]:

Let $x, y, z, tx+ky+lz, ty+kz+lx, tz+kx+ly$ be positive real numbers and let $-k-l < t \le \frac{k+l}{2}$. Then,

$$\frac{x}{tx+ky+lz}+\frac{y}{ty+kz+lx}+\frac{z}{tz+kx+ly}\ge\frac{3}{t+k+l},$$

with equality if and only if $t=k=l$ or $x=y=z$.

Now, let $x=F_n$, $y=F_nL_n$, $z=L_n$, $t=2$, and $k=l=3$. Because

$$-6=-k-l<2=t\le 3=\frac{3+3}{2},$$

it follows that

$$\frac{F_n}{2F_n+3L_n(1+F_n)}+\frac{F_nL_n}{2F_nL_n+3(F_n+L_n)}+\frac{L_n}{2L_n+3F_n(1+L_n)}\ge\frac{3}{2+3+3}=\frac{3}{8},$$

for any positive integer n, with equality if and only if $F_n=L_n=F_nL_n$. This set of conditions holds if and only if $n=1$. Thus, the minimum value of

$$\frac{F_n}{2F_n+3L_n(1+F_n)}+\frac{F_nL_n}{2F_nL_n+3(F_n+L_n)}+\frac{L_n}{2L_n+3F_n(1+L_n)}$$

is $\frac{3}{8}$, and occurs when $n = 1$.

References

[1] S. Wu and O. Furdui, *A note on a conjectured Nesbitt type inequality*, Taiwanese Journal of Mathematics, **15.2** (2011), 449–456.

Also solved by Dmitry Fleischman, Ángel Plaza and Francisco Perdomo (jointly), Hideyuki Ohtsuka, David Stone and John Hawkins (jointly), Nicuşor Zlota, and the proposer.

A p-Radical Inequality

B-1169 Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.
(Vol. 53.2, May 2015)

Let p be a positive integer. Prove that for any integer $n > 1$

$$\frac{F_{n+2}}{2} \le \sqrt[p]{\frac{F_{n+1}^{p+1} - F_n^{p+1}}{(p+1)F_{n-1}}} \le \sqrt[p]{\frac{F_n^p + F_{n+1}^p}{2}}.$$

Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY.

We shall prove a more general result. For any positive real numbers $x \ge y$, and for any positive integer p, we shall prove that

$$\left(\frac{x+y}{p}\right)^p \le \frac{x^{p+1} - y^{p+1}}{(p+1)(x-y)} \le \frac{x^p + y^p}{2}. \tag{1}$$

The desired result will follow if we set $x = F_{n+1}$ and $y = F_n$. It is straightforward to verify that

$$\begin{aligned}
&(p+1)(x^p + y^p) - 2\sum_{i=0}^{p} x^{p-i}y^i \\
&= (p-1)(x^p + y^p) - 2\sum_{i=1}^{p-1} x^{p-i}y^i \\
&= \begin{cases} 2\displaystyle\sum_{i=1}^{\lfloor (p-1)/2 \rfloor} (x^{p-i} - y^{p-i})(x^i - y^i) & \text{if } p \text{ is odd,} \\ (x^{p/2} - y^{p/2})^2 + 2\displaystyle\sum_{i=1}^{\lfloor (p-1)/2 \rfloor} (x^{p-i} - y^{p-i})(x^i - y^i) & \text{if } p \text{ is even.} \end{cases} \\
&\ge 0.
\end{aligned}$$

This proves the second inequality in (1). By setting $t = x/y$, it is clear that in order to establish the first inequality in (1), it suffices to prove that, for any integer $p \ge 1$,

$$f_p(t) = 2^p(t^{p+1} - 1) - (p+1)(t-1)(t+1)^p \ge 0$$

whenever $t \geq 1$. We shall proceed by induction on p. Our claim is valid when $p = 1$ because $f_1(t) = 0$. Assume that $f_p(t) \geq 0$ whenever $t \geq 1$ for some integer $p \geq 1$. We find

$$\begin{aligned} f'_{p+1}(t) &= 2^{p+1}(p+2)t^{p+1} - (p+2)(t+1)^{p+1} - (p+2)(p+1)(t-1)(t+1)^p \\ &= (p+2)[2^p(t^{p+1}-1) - (p+1)(t-1)(t+1)^p + 2^p(t^{p+1}+1) - (t+1)^{p+1}] \\ &= (p+2)[f_p(t) + 2^p(t^{p+1}+1) - (t+1)^{p+1}]. \end{aligned}$$

For $t \geq 1$, we gather from the induction hypothesis that $f_p(t) \geq 0$, and from the power mean inequality we deduce that

$$\sqrt[p+1]{\frac{t^{p+1}+1}{2}} \geq \frac{t+1}{2},$$

which implies that $2^p(t^{p+1}+1) \geq (t+1)^{p+1}$. Hence, $f'_{p+1}(t) \geq 0$, and $f_{p+1}(t)$ is non-decreasing when $t \geq 1$. Together with $f_{p+1}(1) = 0$, we conclude that $f_{p+1}(t) \geq 0$ whenever $t \geq 1$. This completes the induction and the proof of the first inequality in (1).

Also solved by Dmitry Fleischman, Wei-Kai Lei, Hideyuki Ohtsuka, Nicuşor Zlota, and the proposer.

Limits of Radicals Equality

B-1170 Proposed by D. M. Bătineţu–Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School, Buzău, Romania.
(Vol. 53.2, May 2015)

Let $a > 0$, $b > 0$, and $c \geq 0$. Find

(i)

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!aF^c_{n+1}}} - \frac{n^2}{\sqrt[n]{n!bF^c_n}}\right),$$

(ii)

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!aL^c_{n+1}}} - \frac{n^2}{\sqrt[n]{n!bL^c_n}}\right).$$

Solution by Hideyuki Ohtsuka (Saitama, Japan), Ángel Plaza and Francisco Perdomo (Universidad de Las Palmas de Gran Canaria, Spain), Jaroslav Seibert (University of Pardubice, Czech Republic), independently.

Lemma 1. [1] *If the positive sequences $\{p_n\}$ and $\{q_n\}$ are such that*

$$\lim_{n\to\infty}\frac{p_{n+1}}{np_n} = p > 0 \quad \textit{and} \quad \frac{q_n}{p_n} = q > 0,$$

then,

$$\lim_{n\to\infty}(\sqrt[n+1]{p_{n+1}} - \sqrt[n]{q_n}) = \frac{p}{e}\log\frac{e}{q}.$$

(i) *Let*

$$p_n = \frac{n^{2n}}{n!aF_n^c} \quad \textit{and} \quad q_n = \frac{n^{2n}}{n!bF_n^c}.$$

We have

$$\begin{aligned}
\lim_{n\to\infty} \frac{p_{n+1}}{np_n} &= \lim_{n\to\infty} \frac{1}{n} \cdot \frac{(n+1)^{2(n+1)}}{(n+1)!aF_{n+1}^c} \cdot \frac{n!aF_n^c}{n^{2n}} \\
&= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{2n+1} \left(\frac{F_n}{F_{n+1}}\right)^c \\
&= \frac{1}{\alpha^c} \lim_{n\to\infty} \left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^{2n} \\
&= \frac{e^2}{\alpha^c}
\end{aligned}$$

and

$$\frac{q_n}{p_n} = \frac{a}{b}.$$

Using the lemma, we have

$$\begin{aligned}
\lim_{n\to\infty} \left(\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!aF_{n+1}^c}} - \frac{n^2}{\sqrt[n]{n!bF_n^c}} \right) &= \lim_{n\to\infty} \left(\sqrt[n+1]{\frac{(n+1)^{2(n+1)}}{(n+1)!aF_{n+1}^c}} - \sqrt[n]{\frac{n^{2n}}{n!bF_n^c}} \right) \\
&= \lim_{n\to\infty} \left(\sqrt[n+1]{p_{n+1}} - \sqrt[n]{q_n} \right) = \frac{e}{\alpha^c} \log \frac{eb}{a}.
\end{aligned}$$

(ii) *Similarly, we have*

$$\lim_{n\to\infty} \left(\frac{(n+1)^2}{\sqrt[n+1]{(n+1)!aL_{n+1}^c}} - \frac{n^2}{\sqrt[n]{n!bL_n^c}} \right) = \frac{e}{\alpha^c} \log \frac{eb}{a}.$$

References

[1] Gh. Toader, *Lalescu sequences*, Publikacije-Elektrotehnickog Fakulteta Univerzitet U Beogradu Serija Matematika, **9** (1998), 19–28.

Also solved by Dmitry Fleischman and the proposer.

We would like to belatedly acknowledge the solutions to Problems B-1161 through B-1165 by Dmitry Fleischman.