ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
RUSS EULER AND JAWAD SADEK

Please submit all new problem proposals and their solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468, or by email at reuler@nwmissouri.edu. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by May 15, 2015. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1156 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Prove that

$$\sum_{n=0}^{\infty} \tan^{-1}\left(\frac{\sqrt{5}}{L_{4n+2}}\right) = \frac{\pi}{4}.$$

B-1157 **Proposed by Hideyuki Ohtsuka, Saitama, Japan.**

For positive integers a, b, c prove that

$$\frac{F_{2a}}{F_{b+c}} + \frac{F_{2b}}{F_{c+a}} + \frac{F_{2c}}{F_{a+b}} \geq 3.$$

B-1158 **Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School, Buzău, Romania.**

If $a, b, m > 0$, then prove that

$$\sum_{k=1}^{n} \frac{1}{\left(a \cdot F_k^2 + b \cdot \sqrt[n]{\prod_{k=1}^{n} F_k^2}\right)^m} \geq \frac{n^{m+1}}{(a+b)^m F_n^m F_{n+1}^m} \tag{1}$$

and

$$\sum_{k=1}^{n} \frac{1}{\left(a \cdot L_k^2 + b \cdot \sqrt[n]{\prod_{k=1}^{n} L_k^2}\right)^m} \geq \frac{n^{m+1}}{(a+b)^m (L_n L_{n+1} - 2)^m} \tag{2}$$

for any positive integer n.

B-1159 **Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School, Buzău, Romania.**

If $x, y, z > 0$, then prove that

$$\frac{x}{xF_n + yF_{n+1} + zF_{n+2}} + \frac{y}{yF_n + zF_{n+1} + xF_{n+2}} + \frac{z}{zF_n + xF_{n+1} + yF_{n+2}} \geq \frac{3}{2F_{n+2}} \tag{1}$$

and

$$\frac{x}{xL_n + yL_{n+1} + zL_{n+2}} + \frac{y}{yL_n + zL_{n+1} + xL_{n+2}} + \frac{z}{zL_n + xL_{n+1} + yL_{n+2}} \geq \frac{3}{2L_{n+2}} \tag{2}$$

for any positive integer n.

B-1160 **Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School, Buzău, Romania.**

If $e_n = (1 + \frac{1}{n})^n$, then $\lim_{n\to\infty} e_n = e$. Compute each of the following:

(1) $\displaystyle\lim_{n\to\infty} \left(e \cdot \sqrt[n+1]{(n+1)!F_{n+1}} - e_n \cdot \sqrt[n]{n!F_n}\right)$

(2) $\displaystyle\lim_{n\to\infty} \left(e_{n+1} \cdot \sqrt[n+1]{(n+1)!F_{n+1}} - e_n \cdot \sqrt[n]{n!F_n}\right)$

(3) $\displaystyle\lim_{n\to\infty} \left(e \cdot \sqrt[n+1]{(n+1)!L_{n+1}} - e_n \cdot \sqrt[n]{n!L_n}\right)$

(4) $\displaystyle\lim_{n\to\infty} \left(e_{n+1} \cdot \sqrt[n+1]{(n+1)!L_{n+1}} - e_n \cdot \sqrt[n]{n!L_n}\right)$.

SOLUTIONS

Sum of Cubes and the Square of Sum of Cubes

B-1136 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 51.4, November 2013)

Prove that

$$\sum_{k=1}^{n}(F_kF_{k+1})^3 = \left(\sum_{k=1}^{n} F_k^2 F_{k+1}\right)^2.$$

Solution by Zbigniew Jakubczyk, Warsaw, Poland.

We'll prove by induction on n that

$$\sum_{k=1}^{n} F_k^2 F_{k+1} = \frac{F_nF_{n+1}F_{n+2}}{2}.$$

The equation is trivially true for $n = 1$. Suppose that equality is true for a positive integer n. We will show that it is also true for $n + 1$.

$$\begin{aligned}\sum_{k=1}^{n+1} F_k^2 F_{k+1} &= \sum_{k=1}^{n} F_k^2 F_{k+1} + F_{n+1}^2 F_{n+2} \\ &= \frac{F_nF_{n+1}F_{n+2}}{2} + F_{n+1}^2F_{n+2} = \frac{F_{n+1}F_{n+2}(F_n + 2F_{n+2})}{2} \\ &= \frac{F_{n+1}F_{n+2}F_{n+3}}{2}.\end{aligned}$$

Now we want to prove that

$$\sum_{k=1}^{n}(F_kF_{k+1})^3 = \left(\frac{F_nF_{n+1}F_{n+2}}{2}\right)^2.$$

It is easy to see that the equation holds for $n = 1$. Assume that the equation is true for a positive integer n. We will prove that it holds for $n + 1$.

$$\begin{aligned}\sum_{k=1}^{n+1}(F_kF_{k+1})^3 &= \left(\frac{F_nF_{n+1}F_{n+2}}{2}\right)^2 + F_{n+1}^3F_{n+2}^3 \\ &= \frac{F_{n+1}^2F_{n+2}^2(F_n^2 + 4F_{n+1}F_{n+2})}{4} = \frac{F_{n+1}^2F_{n+2}^2(F_{n+2}^2 + F_{n+1}^2 + 2F_{n+1}F_{n+2})}{4} \\ &= \frac{F_{n+1}^2F_{n+2}^2(F_{n+2} + F_{n+1})^2}{4} = \left(\frac{F_{n+1}F_{n+2}F_{n+3}}{2}\right)^2\end{aligned}$$

This completes the proof.

All solvers gave similar solutions.

Also solved by Brian D. Beasley, Kenneth B. Davenport (2 solutions), M. N. Deshpande, Steve Edwards, Dmitry Fleischman, G. C. Greubel, George A. Heisert, Harris Kwong, Victoria A. Landers, Ángel Plaza, Jaroslav Seibert, Jason L. Smith, and the proposer.

By the Binomial Theorem

B-1137 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School, Buzău, Romania.
(Vol. 51.4, November 2013)

Prove that

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\frac{L_m^k+L_{m+1}^k}{L_{m+2}^k}=\frac{L_m^{2n}+L_{m+1}^{2n}}{L_{m+2}^{2n}} \quad \text{for any positive integer } n; \tag{1}$$

and

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\frac{F_m^k+F_{m+1}^k}{F_{m+2}^k}=\frac{F_m^{2n}+F_{m+1}^{2n}}{F_{m+2}^{2n}} \quad \text{for any positive integer } n. \tag{2}$$

Solution by Russell J. Hendel, Towson University and Jaroslav Seibert, Institute of Mathematics, University of Pardubice, Czech Republic (independently).

The formula

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\frac{G_m^k+G_{m+1}^k}{G_{m+2}^k}=\frac{G_m^{2n}+G_{m+1}^{2n}}{G_{m+2}^{2n}}$$

holds for any sequence satisfying the recurrence $G_{n+2}=G_{n+1}+G_n$. We will prove it by using the binomial theorem $(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k}$. In fact,

$$\begin{aligned}
&\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\frac{G_m^k+G_{m+1}^k}{G_{m+2}^k}=\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\frac{G_m^k}{G_{m+2}^k}+\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\frac{G_{m+1}^k}{G_{m+2}^k}\\
&=\sum_{k=0}^{2n}(-1)^{2n-k}\binom{2n}{k}\left(\frac{G_m}{G_{m+2}}\right)^k+\sum_{k=0}^{2n}(-1)^{2n-k}\binom{2n}{k}\left(\frac{G_{m+1}}{G_{m+2}}\right)^k\\
&=\left(\frac{G_m}{G_{m+2}}-1\right)^{2n}+\left(\frac{G_{m+1}}{G_{m+2}}-1\right)^{2n}\\
&=\left(\frac{G_m-G_{m+2}}{G_{m+2}}\right)^{2n}+\left(\frac{G_{m+1}-G_{m+2}}{G_{m+2}}\right)^{2n}\\
&=\left(\frac{-G_{m+1}}{G_{m+2}}\right)^{2n}+\left(\frac{-G_m}{G_{m+2}}\right)^{2n}\\
&=\frac{G_m^{2n}+G_{m+1}^{2n}}{G_{m+2}^{2n}}
\end{aligned}$$

which completes the proof.

Letting $G_n = F_n$ and $G_n = L_n$, we obtain the identities for the Fibonacci and Lucas numbers.

Also solved by Kenneth B. Davenport, M. N. Deshpande, Steve Edwards, Dmitry Fleishman, G. C. Greubel, Zbigniew Jakubczyk (2 solutions), Harris Kwong, Hideyuki Ohtsuka, Ángel Plaza, and the proposer.

A Lower Bound for a Product of Sums

B-1138 Proposed by D. M. Bătineţu–Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School, Buzău, Romania.
(Vol. 51.4, November 2013)

Prove that if $m > 0$ and $p > 0$, then

$$\left(\sum_{k=1}^{n} \frac{F_k^{m+1}}{L_k^m}\right)\left(\sum_{k=1}^{n} \frac{F_k^{p+1}}{L_k^p}\right) \geq \frac{(F_{n+2}-1)^{m+p+2}}{(L_{n+2}-3)^{m+p}},$$

for any positive integer n.

Solution by Kenneth B. Davenport, Dallas, PA.

The approach is very similar to the method employed by Paul Bruckman in his solution of B-1108, Vol. 51.2, p. 181–182. Using a special case of *Holder's Inequality*, for two sequences a_k and b_k, we have

$$\sum_{k=1}^{n} a_k \cdot b_k \leq \left(\sum_{k=1}^{n} a_k^u\right)^{1/u}\left(\sum_{k=1}^{n} b_k^v\right)^{1/v} \tag{1}$$

where $\frac{1}{u}+\frac{1}{v}=1$. We let $a_k = (L_k)^{m/m+1}$ and $b_k = \frac{F_k}{(L_k)^{m/m+1}}$; and we let $\frac{1}{u} = \frac{m}{m+1}$ and $\frac{1}{v} = \frac{1}{m+1}$. Then

$$\sum_{k=1}^{n} a_k \cdot b_k = \sum_{k=1}^{n} F_k = (F_{n+2}-1). \tag{2}$$

Then on the RHS of (1) we have

$$\left(\sum_{k=1}^{n} L_k\right)^{m/m+1}\left(\sum_{k=1}^{n} \frac{F_k^{m+1}}{L_k^m}\right)^{1/m+1}. \tag{3}$$

Raising (2) and (3) to the $(m+1)$ power and further noting $\sum_{k=1}^{n} L_k = L_{n+2}-3$, we get

$$(F_{n+2}-1)^{m+1} \leq (L_{n+2}-3)^m \left(\sum_{k=1}^{n} \frac{F_k^{m+1}}{L_k^m}\right). \tag{4}$$

But now, for another value, say p, we also get the relation

$$(F_{n+2}-1)^{p+1} \leq (L_{n+2}-3)^p \left(\sum_{k=1}^{n} \frac{F_k^{p+1}}{L_k^p}\right). \tag{5}$$

Multiplying (4) by (5) and dividing through by the Lucas sums we obtain the stated inequality. This completes the proof.

All solvers astutely adjusted the proposal to reflect the fact that on the right hand side L_n should be L_{n+2}.

Also solved by Dmitry Fleishman, G. C. Greubel, Hideyuki Ohtsuka, Ángel Plaza, Jaroslav Seibert, and the proposer.

A Lot of Waste

B-1139 Proposed by D. M. Bătineţu–Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.
(Vol. 51.4, November 2013)

Prove that

$$\sum_{k=1}^{n}(1+F_k^2+L_k^2)^2 > 4(F_nF_{n+1}+L_nL_{n+1})-8,$$

for any positive integer n.

Solution by Brian D. Beasley, Presbyterian College, SC.

We show that the summation is not necessary by proving that

$$(1+F_n^2+L_n^2)^2 > 4(F_nF_{n+1}+L_nL_{n+1})-8$$

for any positive integer n. First, we note that this claim holds for $n=1$ and $n=2$. Next, for $n>2$, we have $2L_n > L_{n+1}$ and thus, $L_n^3 \geq 16L_n > 8L_{n+1}$. Then

$$\frac{1}{2}L_n^4 > 4L_nL_{n+1} > 4F_nF_{n+1},$$

so $L_n^4 > 4(F_nF_{n+1}+L_nL_{n+1})$. The claim follows.

Also solved by Charles K. Cook, Kenneth B. Davenport, Dmitry Fleishman, G. C. Greubel, Russell Jay Hendel, Zbigniew Jakubczyk, Harris Kwong, Hideyuki Ohtsuka, Ángel Plaza, Jaroslav Seibert, Rattanapol Wasotharat, and the proposer.

It Adds Up to F_{n+2}

B-1140 Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.
(Vol. 51.4, November 2013)

Let $n \geq 2$ be a positive integer. Show that

$$\frac{1}{2}\left(\frac{F_n^3}{F_{n+1}(F_{n+1}-F_n)}+\frac{F_{n+1}^3}{F_n(F_n-F_{n+1})}+\frac{F_{n+2}^3}{F_{n+1}(F_{n+2}-F_{n+1})}\right)$$

is an integer and determine its value.

Solution by Jessica Booth, Instructor, and Timothy Walker, Graduate Student, Pittsburg State University (jointly).

$$\begin{aligned}
&\frac{1}{2}\left(\frac{F_n^3}{F_{n+1}(F_{n+1}-F_n)}+\frac{F_{n+1}^3}{F_n(F_n-F_{n+1})}+\frac{F_{n+2}^3}{F_{n+1}(F_{n+2}-F_{n+1})}\right)\\
&=\frac{1}{2}\left(\frac{-F_n^3}{F_{n+1}(F_n-F_{n+1})}+\frac{F_{n+1}^3}{F_n(F_n-F_{n+1})}+\frac{F_{n+2}^3}{F_{n+1}(F_{n+2}-F_{n+1})}\right)\\
&=\frac{-F_n^4+F_{n+1}^4+F_{n+2}^3(F_n-F_{n+1})}{2F_nF_{n+1}(F_n-F_{n+1})}\\
&=\frac{(F_{n+1}^2+F_n^2)(F_{n+1}+F_n)(F_{n+1}-F_n)+F_{n+2}^3(F_n-F_{n+1})}{2F_nF_{n+1}(F_n-F_{n+1})}\\
&=\frac{-(F_{n+1}^2+F_n^2)F_{n+2}+F_{n+2}^3}{2F_nF_{n+1}}\\
&=\frac{F_{n+2}(F_{n+2}^2-F_{n+1}^2-F_n^2)}{2F_nF_{n+1}}\\
&=\frac{F_{n+2}(F_n^2+2F_nF_{n+1}+F_{n+1}^2-F_n^2-F_{n+1}^2)}{2F_nF_{n+1}}\\
&=\frac{F_{n+2}(2F_nF_{n+1})}{2F_nF_{n+1}}\\
&=F_{n+2}
\end{aligned}$$

Hence, not only does the equation result in an integer, but also is always F_{n+2}.

Also solved by Brian D. Beasley, Charles K. Cook, Kenneth B. Davenport, Steve Edwards, Dmitry Fleishman, G. C. Greubel, Russell Jay Hendel, Zbigniew Jakubczyk, Harris Kwong, Hideyuki Ohtsuka, Ángel Plaza, Jaroslav Seibert, Rattarapol Wasutharat, and the proposer.