We would like to thank our readers for their submissions during our editorship over the past sixteen years.

Please submit all new problem proposals and their solutions to DR. HARRIS KWONG Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY 14063, or by email at kwong@fredonia.edu.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed (preferably in \LaTeX) on separate sheets. Solutions to problems in this issue must be received by May 15, 2017. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting “well-known results”.

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

### BASIC FORMULAS

The Fibonacci numbers $F_n$ and the Lucas numbers $L_n$ satisfy

\[ F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1; \]
\[ L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1. \]

Also, $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

### PROBLEMS PROPOSED IN THIS ISSUE

**B-1196** Proposed by Hideyuki Ohtsuka, Saitama, Japan.

For any integer $n$, prove that

\[ \frac{L_{n+2}^5 - L_{n+1}^5 - L_n^5 - L_{n-2}^5}{L_{n+2}^3 - L_{n+1}^3 - L_{n-1}^3 - L_{n-2}^3} = 5 \cdot \frac{F_{n+2}^5 - F_{n+1}^5 - F_n^5 - F_{n-2}^5}{F_{n+2}^3 - F_{n+1}^3 - F_{n-1}^3 - F_{n-2}^3}. \]

**B-1197** Proposed by D. M. Bătinețu–Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania.

Let $a$ and $b$ be positive real numbers. For any positive integer $n$, prove each of the following:
(i) \[ \sum_{k=1}^{n} \frac{F_k^4}{a L_k + b F_k^2} > \frac{F_n^2 F_{n+1}^2}{a(L_n+2) - 3 + b F_n F_{n+1}}, \]

(ii) \[ \sum_{k=1}^{n} \frac{F_k^4}{a F_{n+2} + b F_n - a} > \frac{F_n^2 F_{n+1}^2}{(an+b)(F_n^2 - 1)}. \]

**B-1198** Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Let \( c \) be a positive integer. The sequence \( \{a_n\} \) is defined by, for \( n \geq 1 \),

\[ a_{n+2} = a_n + 2c \text{ with } a_1 = 1 \text{ and } a_2 = 3. \]

Prove that

(i) \[ \sum_{n=1}^{\infty} \tan^{-1} \frac{F_c}{a_{n+c}} = \frac{\pi}{4}, \text{ if } c \text{ is even}; \]

(ii) \[ \sum_{n=1}^{\infty} \tan^{-1} \frac{L_c}{a_{n+c}} = \frac{\pi}{4}, \text{ if } c \text{ is odd}. \]

**B-1199** Proposed by Ángel Plaza and Sergio Falcón, Universidad de Las Palmas de Gran Canaria, Spain.

For any positive integer number \( k \), the \( k \)-Fibonacci and \( k \)-Lucas sequences, say \( \{F_{k,n}\}_{n \in \mathbb{N}} \) and \( \{L_{k,n}\}_{n \in \mathbb{N}} \) both are defined recursively by \( u_{n+1} = ku_n + u_{n-1} \) for \( n \geq 1 \), with respective initial conditions \( F_{k,0} = 0; F_{k,1} = 1 \) and \( L_{k,0} = 2; F_{k,1} = k \). Prove that for all integers \( m \geq 1 \) and \( r \geq 1 \),

\[ k r^{m+1} F_{k,m+1} = \sum_{i=0}^{m} r^i L_{k,i} + (kr-2) \sum_{i=0}^{m+1} r^{i-1} F_{k,i}. \]

**B-1200** Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Let \( H_n \) denote the \( n \)th harmonic number. Prove that

(i) \[ \sum_{n=1}^{\infty} \frac{H_n F_n}{2^n} = \log \left( \frac{4 \alpha^{12}/\sqrt{5}}{\phi} \right) \] and (ii) \[ \sum_{n=1}^{\infty} \frac{H_n L_n}{2^n} = \log \left( \frac{64 \alpha^{4\sqrt{5}}}{\phi} \right). \]
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SOLUTIONS

Closure !

B-1176 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 53.4, November 2015)

Find a closed form expression for the sum
\[ \sum_{k=0}^{n} L_{3k}^3 + 2 \sum_{k=0}^{n} L_{3k}. \]

Solution by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

Working from the Binet formula for \( L_n \), \( L_n = \alpha^n + \beta^n \), we find
\[ L_{3k}^3 = \alpha^{3n} + \beta^{3n} + 3(\alpha\beta)^n(\alpha^n + \beta^n) = L_{3n} + 3(-1)^n L_n. \]
Therefore,
\[ L_{3k} = L_{3k+1} + 3(-1)^k L_{3k} = L_{3k+1} - 3L_{3k}, \]
and
\[ \sum_{k=0}^{n} L_{3k}^3 + 2 \sum_{k=0}^{n} L_{3k} = \sum_{k=0}^{n} L_{3k+1} - \sum_{k=0}^{n} L_{3k} = L_{3n+1} - L_1 = L_{3n+1} - 1. \]


Find Their Limits

B-1177 Proposed by Ángel Plaza and Sergio Falcón, Universidad de Las Palmas de Gran Canaria, Spain.
(Vol. 53.4, November 2015)

For any integer \( p \geq 0 \), prove each of the following:

(i) \[ \lim_{n \to \infty} \frac{F_n^p + F_{n+3}^p}{F_{n+2}^p} = \frac{L_{2p} + L_p}{2} - \frac{F_{2p} - F_p}{2} \sqrt{5}, \]

(ii) \[ \lim_{n \to \infty} \frac{L_n^p + L_{n+3}^p}{L_{n+2}^p} = \frac{L_{2p} + L_p}{2} - \frac{F_{2p} - F_p}{2} \sqrt{5}. \]
Solution by Jaroslav Seibert, Institute of Mathematics and Quantitative Methods, Faculty of Economics and Administration, University of Pardubice, Czech Republic.

We will consider the generalized Fibonacci numbers defined by the recurrence relation

\[ G_{n+2} = G_{n+1} + G_n \]

with arbitrary initial terms \( G_0, G_1 \).

It is known that \( \lim_{n \to \infty} \frac{G_{n+k}}{G_n} = \alpha^k \) (e.g.\([1, \text{p. } 247]\)) for each integer \( k \). Then the following holds for the limit on the left-hand side of the given equalities

\[
\lim_{n \to \infty} \frac{G_p}{G_{n+2}^p} + \frac{G_p^p}{G_{n+2}^p} = \lim_{n \to \infty} \left( \frac{G_n}{G_{n+2}} \right)^p + \lim_{n \to \infty} \left( \frac{G_{n+3}}{G_{n+2}} \right)^p = \left( \frac{1}{\alpha^2} \right)^p + \alpha^p = \frac{1}{\alpha^{2p}} + \alpha^p = \frac{1 + \alpha^{3p}}{\alpha^{2p}}.
\]

The expressions on the right-hand side of (i) and (ii) can be rewritten using the Binet formula and the relation \( \alpha \beta = -1 \), as

\[
\frac{L_{2p} + L_p}{2} - \frac{F_{2p} - F_p}{2} \sqrt{5} = \frac{\alpha^{2p} + \beta^{2p} + \alpha^p + \beta^p}{2} - \frac{\alpha^{2p} - \beta^{2p} - \alpha^p + \beta^p}{2 \sqrt{5}} \sqrt{5}
\]

\[
= \beta^{2p} + \alpha^p
\]

\[
= \left( -\frac{1}{\alpha} \right)^{2p} + \alpha^p
\]

\[
= \frac{1}{\alpha^{2p}} + \alpha^p
\]

\[
= 1 + \alpha^{3p}
\]

\[
= \frac{1 + \alpha^{3p}}{\alpha^{2p}}.
\]

It follows that the equality \( \lim_{n \to \infty} \frac{G_p}{G_{n+2}^p} + \frac{G_p^p}{G_{n+2}^p} = \frac{L_{2p} + L_p}{2} - \frac{F_{2p} - F_p}{2} \sqrt{5} \) holds for the generalized Fibonacci numbers \( G_n \), and in particular, for the Fibonacci and Lucas numbers.

References


Also solved by Adnan Ali (student), Brian D. Beasley, Brian Bradie, Miguel Cidras, Charlie K. Cook and Michael R. Bacon (jointly), Kenneth B. Davenport, Steve Edwards, Dmitry Fleischman, John Hawkins and David Stone (jointly), Hideyuki Ohtsukas, Dan Weiner, and the proposer.

The Ubiquitous AM-GM Inequality

B-1178 Proposed by D. M. Bătinețu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School, Buzău, Romania.

(Vol. 53.4, November 2015)

Prove that

\[
\sqrt{F_1^4 - F_2^2 F_2^2} + F_3 \sqrt{F_3^4 - F_2^2 F_3^2} + \cdots + \sqrt{F_{n-1}^4 - F_{n-1}^2 F_n^2} + F_n + \sqrt{F_n^4 - F_n^2} + 1 > F_n F_{n+1}.
\]
Given any real numbers $x$ and $y$, the AM-GM inequality asserts that $x^4 + y^4 \geq 2x^2y^2$. Then

$$4(x^4 - x^2y^2 + y^4) = x^4 + y^4 + 3(x^4 + y^4) - 4x^2y^2$$
$$\geq x^4 + y^4 + 6x^2y^2 - 4x^2y^2$$
$$= (x^2 + y^2)^2.$$ 

Hence,

$$\sqrt{x^4 - x^2y^2 + y^4} \geq \frac{x^2 + y^2}{2},$$

with equality holds only when $x^2 = y^2$. Take note that $F_n^4 - F_n^2 + 1 = F_n^4 - F_n^2 F_1^2 + F_1^4$. The sum $S$ on the left-hand side of the given inequality is a cyclic sum. Therefore,

$$S > \sum_{i=1}^{n} F_i^2 = F_nF_{n+1}.$$ 

Remark: It is clear that the argument can also be applied to the Lucas numbers. Since $L_1 = 1$, and $\sum_{i=1}^{n} L_i^2 = L_nL_{n+1} - 2$, we find the analogous result

$$\sqrt{L_1^4 - L_1^2 L_2^2 + L_2^4} + \sqrt{L_2^4 - L_2^2 L_3^2 + L_3^4} + \cdots + \sqrt{L_{n-1}^4 - L_{n-1}^2 L_n^2 + L_n^4} + \sqrt{L_n^4 - L_n^2 + 1} > L_nL_{n+1} - 2.$$ 

Also solved by Adnan Ali (student), Kenneth B. Davenport, Dmitry Fleischman, Hideyuki Ohtsuka, Angel Plaza, Jaroslav Seibert, Itzal Urioste, Nicușor Zlota, and the proposer.

**Sine, Cosine, and a Fibonacci Inequality**

**B-1179** Proposed by D. M. Bătinețu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade School, Buzău, Romania.

(Vol. 53.4, November 2015)

Let $\{x_n\}_{n \geq 1}$ be a sequence of real numbers. Prove that

$$2 \cdot \left( \sum_{k=1}^{n} F_k \cdot \sin x_k \right) \cdot \left( \sum_{k=1}^{n} F_k \cdot \cos x_k \right) \leq n \cdot F_n \cdot F_{n+1}.$$ 

From the AM-GM and Cauchy-Schwartz Inequalities, we have
\[
2 \cdot \left( \sum_{k=1}^{n} F_k \cdot \sin x_k \right) \cdot \left( \sum_{k=1}^{n} F_k \cdot \cos x_k \right) \leq \left( \sum_{k=1}^{n} F_k \cdot \sin x_k \right)^2 + \left( \sum_{k=1}^{n} F_k \cdot \cos x_k \right)^2 \\
\leq \left( \sum_{k=1}^{n} F_k^2 \right) \left( \sum_{k=1}^{n} \sin^2 x_k \right) + \left( \sum_{k=1}^{n} F_k^2 \right) \left( \sum_{k=1}^{n} \cos^2 x_k \right) \\
= n \left( \sum_{k=1}^{n} F_k^2 \right) n \cdot F_n \cdot F_{n+1}.
\]

Also solved by Brian Bradie, Miguel Cidras, Kenneth B. Davenport, Steve Edwards, Dmitry Fleischman, Harris Kwong, Wei-Kai Lai, Hideyuki Ohtsuka, Ángel Plaza, Henri RiCardo, Jaroslav Seibert, Itzal de Urioste, Dan Weiner, Nicușor Zlota, and the proposer.

**A Complex One!**

B-1180  Hideyuki Ohtsuka, Saitama, Japan.  
(Vol. 53.4, November 2015)

If \( i = \sqrt{-1} \), find
\[
\sum_{n=0}^{\infty} \frac{1}{F_{2n} + i}.
\]

Solution by Steve Edwards, Kennesaw State University, Marietta, GA.

We use the identities \( F_{2n}^2 + 1 = F_{2n-1}F_{2n+1} \) and \( F_{2n+2}F_{2n-1} - F_{2n}F_{2n+1} = 1 \), which can be found in [1].

For the summand we have
\[
\frac{1}{F_{2n} + i} = \frac{1}{F_{2n}} \cdot \frac{F_{2n} - i}{F_{2n} + i} = \frac{F_{2n} - i}{F_{2n}^2 + 1} = \frac{F_{2n+1} - F_{2n-1}}{F_{2n-1}F_{2n+1} - iF_{2n+2}F_{2n-1} - F_{2n}F_{2n+1}} = \left( \frac{1}{F_{2n-1}} - \frac{1}{F_{2n+1}} \right) - i \left( \frac{F_{2n+2}}{F_{2n+1}} - \frac{F_{2n}}{F_{2n-1}} \right).
\]

The sum is telescoping, and the \( k \)th partial sum is
\[
\left( \frac{1}{F_{-1}} - \frac{1}{F_{2k+1}} \right) - i \left( \frac{F_{2k+2}}{F_{2k+1}} - \frac{F_{0}}{F_{1}} \right),
\]

and since \( \lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \alpha \), the series converges to \( 1 - \alpha i \).
References


Also solved by Adnan Ali (student), Brian Beasley, Brian Bradie, Kenneth B. Davenport, Dmitry Fleischman, Harris Kwong, Poo-Sung Park, Ángel Plaza, Jaroslav Seibert, Dan Weiner, and the proposer.

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