ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY
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Please submit solutions and problem proposals to Dr. Harris Kwong, Department of Mathematical Sciences, SUNY Fredonia, Fredonia, NY, 14063, or by email at kwong@fredonia.edu. If you wish to have receipt of your submission acknowledged by mail, please include a self-addressed, stamped envelope.

Each problem or solution should be typed on separate sheets. Solutions to problems in this issue must be received by November 15, 2017. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results."

The content of the problem sections of The Fibonacci Quarterly are all available on the web free of charge at www.fq.math.ca/.

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1206 Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.

Let $n \geq 2$ be an integer. Prove that

$$1 + \frac{1}{n^2} \sum_{1 \leq i < j \leq n} \frac{\left(\sqrt{F_i F_{j+1}} - \sqrt{F_{i+1} F_j}\right)^2}{F_i F_j} \leq \frac{1}{n} \sum_{k=1}^{n} \frac{F_{k+1}}{F_k},$$

in which the subscripts are taken modulo n.

B-1207 Proposed by D. M. Bătineţu-Giurgiu, Matei Basarab National College, Bucharest, Romania, and Neculai Stanciu, George Emil Palade School, Buzău, Romania.

Prove that

$$\frac{F_n^4 + F_1^4}{F_n^2 + F_1^2} + \sum_{k=1}^{n-1} \frac{F_k^4 + F_{k+1}^4}{F_{2k+1}} > F_n F_{n+1}$$

for any integer $n > 1$.

B-1208 Proposed by Ivan V. Fedak, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

For every positive integer n, find all real solutions of the following linear system of equations:

$$\begin{array}{ccccccccccccccc}
F_1x_1 & + & x_2 & & & & & & & & & & & = & F_3,\\
F_2x_1 & + & F_1x_2 & + & x_3 & & & & & & & & & = & F_4,\\
F_3x_1 & + & F_2x_2 & + & F_1x_3 & + & \cdots & & & & & & & = & F_5,\\
\vdots & & \vdots & & \vdots & & & \ddots & & & & & & & \vdots\\
F_{n-1}x_1 & + & F_{n-2}x_2 & + & F_{n-3}x_3 & + & \cdots & + & x_n & & & & & = & F_{n+1},\\
F_nx_1 & + & F_{n-1}x_2 & + & F_{n-2}x_3 & + & \cdots & + & F_1x_n & + & x_{n+1} & & & = & F_{n+2},\\
F_{n+1}x_1 & + & F_nx_2 & + & F_{n-1}x_3 & + & \cdots & + & F_2x_n & + & F_1x_{n+1} & & & = & F_{n+3}-1.
\end{array}$$

B-1209 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

The Tribonacci numbers T_n satisfy $T_0 = 0$, $T_1 = T_2 = 1$, and

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \quad \text{for } n \geq 3.$$

Prove that

$$\sum_{k=1}^{n} T_{2k}T_{2k-1} = \left(\sum_{k=1}^{n} T_{2k-1}\right)^2$$

for any integer $n \geq 1$.

B-1210 Proposed by Taras Goy, Vasyl Stefanyk Precarpathian National University, Ivano-Frankivsk, Ukraine.

Prove that

$$\sum_{t_1+2t_2+\cdots+nt_n=n} (-1)^{n-s} \frac{s!}{t_1!\,t_2!\,\cdots\,t_n!} F_1^{t_1}F_2^{t_2}\cdots F_n^{t_n} = \frac{1-(-1)^n}{2},$$

where $s = t_1 + t_2 + \cdots + t_n$.

SOLUTIONS

A Telescoping Lucas Sum

B-1186 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 54.2, May 2016)

Prove that

$$\sum_{n=0}^{\infty} \frac{(-1)^n L_{2^n}}{L_{2^{n+1}}+1} = 0.$$

Solution by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

Since $L_{2k} + 2(-1)^k = L_k^2$, for any even number m, we have $L_{2m} = L_m^2 - 2$. This applies to every denominator in the given expression for $n > 0$, and hence,

$$\frac{(-1)^n L_{2^n}}{L_{2^{n+1}}+1} = \frac{(-1)^n L_{2^n}}{L_{2^n}^2 - 1} = \frac{(-1)^n (L_{2^n} - 1 + 1)}{(L_{2^n}-1)(L_{2^n}+1)} = \frac{(-1)^n}{L_{2^n}+1} + \frac{(-1)^n}{L_{2^{n+1}}+1}.$$

Therefore, the given sum telescopes, from the second term; thus,

$$\sum_{n=0}^{N} \frac{(-1)^n L_{2^n}}{L_{2^{n+1}}+1} = \frac{(-1)^N}{L_{2^{N+1}}+1},$$

from which the result follows.

Also solved by Jeremiah Bartz, Brian Bradie, Kenny B. Davenport, Steve Edwards, I. V. Fedak, Dmitry G. Fleischman, G. C. Greubel, Sai Gopal Rayaguru, Jaroslav Seibert, David Terr, Dan Weiner, and the proposer.

The Same Fibonacci Number

B-1187 Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain. (Vol. 54.2, May 2016)

Let $n \geq 1$ be a positive integer. Find all real solutions of the following system of equations:

$$x^3 + L_n x + y = F_n(1+L_n) + F_n x^2,$$
$$F_n y^3 + F_{2n} y + z = F_n(1+F_{2n}) + F_n^2 y^2,$$
$$L_n z^3 + L_n^2 z + x = F_n(1+L_n^2) + F_{2n} z^2.$$

Solution by Brian D. Beasley, Presbyterian College, Clinton, SC.

We use the identity $F_{2n} = F_n L_n$ to rewrite the system as

$$(x^2+L_n)(x-F_n) = F_n - y,$$
$$F_n(y^2+L_n)(y-F_n) = F_n - z,$$
$$L_n(z^2+L_n)(z-F_n) = F_n - x.$$

Then substitution yields

$$P \cdot (x - F_n) = F_n - x,$$

where

$$P = L_n(z^2+L_n) \cdot F_n(y^2+L_n)(x^2+L_n).$$

Since $n \geq 1$ and x, y, and z are real, we have $P > 0$, and hence, $x = F_n$. Thus, $y = F_n$ and $z = F_n$ as well. Note that the result also holds when $n = 0$, as the unique solution of the system in that case is $x = y = z = 0 = F_0$.

Also solved by I. V. Fedak, Dmitry Fleischman, G. C. Greubel, and the proposer.

A Telescoping Fibonacci Sum

B-1188 Proposed by Kenny B. Davenport, Dallas, PA. (Vol. 54.2, May 2016)

Find a closed form expression for

$$5\sum_{k=0}^{n} F_{3^k}^2 - 4\sum_{k=0}^{n} F_{3^k}.$$

Solution by Brian Bradie, Christopher Newport University, Newport News, VA.

Working from the Binet formula for F_n,

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}},$$

we find

$$F_n^3 = \frac{\alpha^{3n} - \beta^{3n} - 3(\alpha\beta)^n(\alpha^n - \beta^n)}{5\sqrt{5}} = \frac{F_{3n} - 3(-1)^n F_n}{5}.$$

Therefore,

$$5F_{3^k}^3 = F_{3^{k+1}} - 3(-1)^{3^k} F_{3^k} = F_{3^{k+1}} + 3F_{3^k},$$

and

$$5\sum_{k=0}^{n} F_{3^k}^2 - 4\sum_{k=0}^{n} F_{3^k} = \sum_{k=0}^{n} F_{3^{k+1}} - \sum_{k=0}^{n} F_{3^k} = F_{3^{n+1}} - F_1 = F_{3^{n+1}} - 1.$$

Editor's Remark: A similar result for the Lucas numbers,

$$\sum_{k=0}^{n} L_{3^k}^2 + 2\sum_{k=0}^{n} L_{3^k} = L_{3^{n+1}} - L_1 = L_{3^{n+1}} - 1,$$

appeared as Elementary Problem B-1176 in Volume 53.4 (November 2015).

Also solved by Itzal De Urioste (student), Steve Edwards, I. V. Fedak, Dmitry Fleischman, G. C. Greubel, Harris Kwong, Hideyuki Ohtsuka, Ángel Plaza, Sai Gopal Rayaguru, Jaroslav Seibert, David Terr, Dan Weiner, and the proposer.

Our Old Friend Binomial Theorem

B-1189 Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.
(Vol. 54.2, May 2016)

Find a closed form for

$$\sum_{k=0}^{2n} \binom{2n}{k} L_{2n-2k}.$$

Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY.

We shall derive a general result. Take note that, for any nonnegative integer m,

$$\sum_{k=0}^{m} \binom{m}{k} \alpha^{m-2k} = \sum_{k=0}^{m} \binom{m}{k} \alpha^{m-k}(\alpha^{-1})^k = (\alpha + \alpha^{-1})^m = (\alpha - \beta)^m.$$

In a similar manner, we also find

$$\sum_{k=0}^{m} \binom{m}{k} \beta^{m-2k} = (\beta - \alpha)^m.$$

Therefore,

$$\sum_{k=0}^{m} \binom{m}{k} L_{m-2k} = \sum_{k=0}^{m} \binom{m}{k} (\alpha^{m-2k} + \beta^{m-2k}) = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ 2 \cdot 5^{m/2} & \text{if } m \text{ is even.} \end{cases}$$

Editor's Remark: G. C. Greubel noted that $\sum_{k=0}^{2n}\binom{2n}{k}F_{2n-2k}=0$. Using the same argument shown above, we see that, for any nonnegative integer m,

$$\sum_{k=0}^{m}\binom{m}{k}F_{m-2k}=\sum_{k=0}^{m}\binom{m}{k}\frac{\alpha^{m-2k}-\beta^{m-2k}}{\sqrt{5}}=\begin{cases}2\cdot 5^{(m-1)/2} & \text{if } m \text{ is odd,}\\ 0 & \text{if } m \text{ is even.}\end{cases}$$

Also solved by Brian Bradie, Kenny B. Davenport, Itzal De Urioste (student), Steve Edwards, I. V. Fedak, Dmitry Fleischman, G. C. Greubel, Ralph P. Grimaldi, Russell Jay Hendel, Hideyuki Ohtsuka, Sai Gopal Rayaguru, Jaroslav Seibert, Jason L. Smith, David Terr, and the proposer.

A Cyclic Sum

B-1190 Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain. (Vol. 54.2, May 2016)

Let $n \ge 1$ be a positive integer. Compute

$$\frac{F_{n+2}}{F_nF_{n+1}}\left(\frac{F_n^n+F_{n+1}^n-F_{n+2}^n}{F_n^{n-1}+F_{n+1}^{n-1}+F_{n+2}^{n-1}}\right)+\frac{F_{n+3}}{F_{n+1}F_{n+2}}\left(\frac{F_{n+1}^n+F_{n+2}^n-F_n^n}{F_n^{n-1}+F_{n+1}^{n-1}+F_{n+2}^{n-1}}\right)$$
$$+\frac{2F_n+F_{n+1}}{F_{n+2}F_n}\left(\frac{F_{n+2}^n+F_n^n-F_{n+1}^n}{F_n^{n-1}+F_{n+1}^{n-1}+F_{n+2}^{n-1}}\right).$$

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA.

Because

$$\frac{F_{n+2}}{F_nF_{n+1}}=\frac{F_{n+1}+F_n}{F_nF_{n+1}}=\frac{1}{F_n}+\frac{1}{F_{n+1}},$$
$$\frac{F_{n+3}}{F_{n+1}F_{n+2}}=\frac{F_{n+2}+F_{n+1}}{F_{n+1}F_{n+2}}=\frac{1}{F_{n+1}}+\frac{1}{F_{n+2}},$$
$$\frac{2F_n+F_{n+1}}{F_{n+1}F_{n+2}}=\frac{F_n+F_{n+2}}{F_{n+2}F_n}=\frac{1}{F_{n+2}}+\frac{1}{F_n},$$

it follows that

$$\frac{F_{n+2}}{F_nF_{n+1}}\left(F_n^n+F_{n+1}^n-F_{n+2}^n\right)$$
$$=F_n^{n-1}+F_{n+1}^{n-1}+\frac{1}{F_n}\left(F_{n+1}^n-F_{n+2}^n\right)+\frac{1}{F_{n+1}}\left(F_n^n-F_{n+2}^n\right),$$
$$\frac{F_{n+3}}{F_{n+1}F_{n+2}}\left(F_{n+1}^n+F_{n+2}^n-F_n^n\right)$$
$$=F_{n+1}^{n-1}+F_{n+2}^{n-1}+\frac{1}{F_{n+1}}\left(F_{n+2}^n-F_n^n\right)+\frac{1}{F_{n+2}}\left(F_{n+1}^n-F_n^n\right),$$
$$\frac{2F_n+F_{n+1}}{F_{n+2}F_n}\left(F_{n+2}^n+F_n^n-F_{n+1}^n\right)$$
$$=F_{n+2}^{n-1}+F_n^{n-1}+\frac{1}{F_{n+2}}\left(F_n^n-F_{n+1}^n\right)+\frac{1}{F_n}\left(F_{n+2}^n-F_{n+1}^n\right).$$

Therefore,

$$\frac{F_{n+2}}{F_nF_{n+1}}\left(\frac{F_n^n+F_{n+1}^n-F_{n+2}^n}{F_n^{n-1}+F_{n+1}^{n-1}+F_{n+2}^{n-1}}\right)+\frac{F_{n+3}}{F_{n+1}F_{n+2}}\left(\frac{F_{n+1}^n+F_{n+2}^n-F_n^n}{F_n^{n-1}+F_{n+1}^{n-1}+F_{n+2}^{n-1}}\right)$$
$$+\frac{2F_n+F_{n+1}}{F_{n+2}F_n}\left(\frac{F_{n+2}^n+F_n^n-F_{n+1}^n}{F_n^{n-1}+F_{n+1}^{n-1}+F_{n+2}^{n-1}}\right)=\frac{2(F_n^{n-1}+F_{n+1}^{n-1}+F_{n+2}^{n-1})}{F_n^{n-1}+F_{n+1}^{n-1}+F_{n+2}^{n-1}}=2.$$

Solution 2 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

The result is 2 for all integers $n \geq 1$. Let us consider the following more general expression

$$E=\frac{1}{a^{n-1}+b^{n-1}+c^{n-1}}\sum_{\text{cyclic}}\frac{a+b}{ab}\,(a^n+b^n-c^n).$$

Notice that

$$\begin{aligned}\sum_{\text{cyclic}}\frac{a+b}{ab}\,(a^n+b^n-c^n)&=\frac{1}{abc}\sum_{\text{cyclic}}c(a+b)(a^n+b^n-c^n)\\&=\frac{1}{abc}\sum_{\text{cyclic}}[c(a^{n+1}+b^{n+1})-(a+b)c^{n+1}+abc(a^{n-1}+b^{n-1})]\\&=\frac{2abc(a^{n-1}+b^{n-1}+c^{n-1})}{abc}\\&=2(a^{n-1}+b^{n-1}+c^{n-1}),\end{aligned}$$

from which we find $E=2$.

Also solved by Kenny B. Davenport, Itzal De Urioste (student), Steve Edwards, I. V. Fedak, Dmitry G. Fleischman, G. C. Greubel, Marcus Harbol and Luke Tiscareno (students), George A. Hisert, Wei-Kai Lai and John Risher (student) (jointly), Hideyki Ohtsuka, Sai Gopal Rayaguru, Jaroslav Seibert, Jason L. Smith, and the proposer.

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