ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by August 15, 2006. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n,\ F_0 = 0,\ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n,\ L_0 = 2,\ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2,\ \beta = (1-\sqrt{5})/2,\ F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1010 Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, MA

Find positive integers $a,\ b$, and m (with $m > 1$) such that

$$F_n \equiv b^n - a^n \pmod{m}$$

is an identity (i.e., true for all n) or prove that no identity of this form exists.

B-1011 Proposed by José Luis Díaz-Barrero, Universidad Politécnica de Cataluña, Barcelona, Spain

Let n be a positive integer. Prove that

$$F_n^5 + L_n^5 = 2F_{n+1}^2(16F_{n+1}^3 - 20F_{n+1}F_{2n} + 5F_{2n}).$$

B-1012 Proposed by Br. J. Mahon, Australia

Prove that

$$\sum_{i=1}^{n} \tan^{-1} \frac{1}{F_{2i-1}} = \tan^{-1} F_{2n}.$$

B-1013 Proposed by the Solution Editor

Prove that

$$\frac{F_n^{2^n} F_{n+1}^{2^n}}{n^{2^n-1}} \leq F_1^{2^{n+1}} + F_2^{2^{n+1}} + \cdots + F_n^{2^{n+1}}$$

for all integers $n \geq 1$.

B-1014 Proposed by Steve Edwards, Southern Polytechnic State University, Marietta, GA

Show that $\sum_{k=0}^{\infty} \alpha^{-nk}$ equals $\frac{F_n\alpha+F_{n+1}-1}{L_n-2}$ if n is a positive even integer, and $\frac{F_n\alpha+F_{n+1}+1}{L_{n+2}}$ if n is a positive odd integer.

B-1015 Proposed by José Luis Díaz-Barrero and Miquel Grau-Sánchez, Universidad Politécnica de Cataluña, Barcelona, Spain

Let n be a positive integer. Prove that

$$\left(\sum_{k=1}^{n} F_k F_{2k}\right) \left(\sum_{k=1}^{n} \frac{F_k^2}{\sqrt{L_k}}\right)^2 \leq F_n^3 F_{n+1}^3.$$

SOLUTIONS

Two Lucas Congruences

B-996 Proposed by Paul S. Bruckman, Canada (Vol. 43, no. 2, May 2005)

Prove the following congruences, for all integers n:

(1) $L_n \equiv 30^n + 50^n \pmod{79}$;

(2) $L_n \equiv 10^n + 80^n \pmod{89}$.

Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY

Noting that $30 \cdot 50 \equiv -1 \pmod{79}$ and that 79 is prime, we find the zeros of

$$q^2 - q - 1 \equiv q^2 - 80q + 30 \cdot 50 = (q-30)(q-50) \pmod{79}$$

to be 30 and 50. Thus

$$L_n \equiv A \cdot 30^n + B \cdot 50^n \pmod{79}$$

for some constants A and B. The initial values $L_0 = 2$ and $L_1 = 1$ yield a system of congruences:

$$\begin{array}{rcrcll} A & + & B & \equiv & 2 & \pmod{79}, \\ 30A & + & 50B & \equiv & 1 & \pmod{79}. \end{array}$$

Eliminating B gives $20A \equiv 99 \equiv 20 \pmod{79}$, hence $A \equiv 1 \pmod{79}$. Likewise, $20B \equiv -59 \equiv 20 \pmod{79}$, thus $B \equiv 1 \pmod{79}$. Therefore

$$L_n \equiv 30^n + 50^n \pmod{79}.$$

The proof of (2) follows a similar argument, and is omitted here.

James A. Sellers extended the result to negative subscripts and H.-J. Seiffert proved, more generally, that if p and q are any integers such that $p + q \neq 1$ and $pq \equiv -1 \pmod{p+q-1}$, then, for all integers n, $L_n \equiv p^n + q^n \pmod{p+q-1}$.

Also solved by Ovidiu Furdui, G.C. Greubel, Ralph P. Grimaldi, Joseph Koštál, H.-J. Seiffert, James A. Sellers, Pavel Trojovský, and the proposer.

Simplify the Sum

B-997 Proposed by Br. J. Mahon, Australia (Vol. 43, no. 2, May 2005)

Prove that

$$\sum_{i=1}^{n} \frac{L_{i-3} - 5(-1)^i}{(L_i + 1)(L_{i+1} + 1)} = \frac{3}{2} - \frac{L_n + 1}{L_{n+1} + 1}$$

for all $n \geq 1$.

Solution by José Luis Díaz-Barrero, UPC, Barcelona, Spain

We claim that

$$L_{i-3} - 5(-1)^i = (1 + L_{i-1})(1 + L_{i+1}) - (1 + L_i)^2 \tag{1}$$

from which immediately follows

$$\frac{L_{i-3} - 5(-1)^i}{(L_i + 1)(L_{i+1} + 1)} = \frac{(1 + L_{i-1})(1 + L_{i+1}) - (1 + L_i)^2}{(L_i + 1)(L_{i+1} + 1)}$$

$$= \frac{L_{i-1} + 1}{L_i + 1} - \frac{L_i + 1}{L_{i+1} + 1}.$$

Therefore,

$$\sum_{i=1}^{n} \frac{l_{i-3} - 5(-1)^i}{(L_i + 1)(L_{i+1} + 1)} = \sum_{i=1}^{n} \left(\frac{L_{i-1} + 1}{L_i + 1} - \frac{L_i + 1}{L_{i+1} + 1} \right) = \frac{3}{2} - \frac{L_n + 1}{L_{n+1} + 1}.$$

To prove (1), we write it in the most convenient form

$$L_{i-3} - 5(-1)^i = L_{i+1} + L_{i-1} + L_{i+1}L_{i-1} - 2L_i - L_i^2.$$

Using Binet's formula, it is straight forward to show

$$L_{i+1} + L_{i-1} + L_{i+1}L_{i-1} - 2L_i - L_i^2 = L_{i-3} - 5(-1)^i$$

and we are done.

Also solved by Paul S. Bruckman, Kenneth B. Davenport, Steve Edwards, Ovidiu Furdui, G.C. Greubel, Ralph P. Grimaldi, Emrah Kilic, Harris Kwong, H. -J. Seiffert, James A. Sellers, Paval Trojovský, and the proposer.

Easier Than How It Looks!

B-998 Proposed by José Luis Díaz-Barrero, Universitat Politécnica de Catalunya, Barcelona, Spain (Vol. 43, no. 2, May 2005)

Let n be a positive integer and let $F_n,\ L_n$ and P_n be respectively the n^{th} Fibonacci, Lucas and Pell number. Prove that

$$\frac{\left| \frac{|F_n - L_n|}{F_{2n}} + \frac{2F_{n+1}}{F_{2n}} - \frac{2}{P_n} \right| + \frac{|F_n - L_n|}{F_{2n}} + \frac{2F_{n+1}}{F_{2n}} + \frac{2}{P_n}}{\max\left\{ \frac{1}{F_n}, \frac{1}{L_n}, \frac{1}{P_n} \right\}}$$

is an integer and determine its value.

Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY

Denote the given quotient r_n. We have $r_1 = r_2 = 4$, hence we shall assume $n \geq 3$. It is easy to verify that $F_n \leq L_n \leq P_n$. Hence $\max\{\frac{1}{F_n}, \frac{1}{L_n}, \frac{1}{P_n}\} = \frac{1}{F_n}$, and

$$\frac{|F_n - L_n|}{F_{2n}} + \frac{2F_{n+1}}{F_{2n}} = \frac{L_n - F_n + 2F_{n+1}}{F_{2n}} = \frac{L_n + F_{n-1} + F_{n+1}}{F_{2n}} = \frac{2L_n}{F_n L_n} = \frac{2}{F_n} \geq \frac{2}{P_n}.$$

Thus

$$\left|\frac{|F_n - L_n|}{F_{2n}} + \frac{2F_{n+1}}{F_{2n}} - \frac{2}{P_n}\right| + \frac{|F_n - L_n|}{F_{2n}} + \frac{2F_{n+1}}{F_{2n}} + \frac{2}{P_n} = \frac{4}{F_n},$$

which leads to the conclusion that $r_n = 4$ for all positive integers n.

Also solved by Paul S. Bruckman, Charles K. Cook, Ovidiu Furdui, George C. Greubel, Emrah Kilic, H. -J. Seiffert, Pavel Trojovský, and the proposer.

Fibonacci Exponentiated

B-999 Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI (Vol. 43, no. 2, May 2005)

Prove that

$$e^{2\sum_{k=1}^{n} \frac{F_{k-1}}{F_{k+2}}} \leq F_{n+1} \leq e^{\sum_{k=1}^{n} \frac{F_{k-1}}{\sqrt{F_k F_{k+1}}}}$$

for all $n \geq 1$.

Solution by H.-J. Seiffert, Thorwaldsenstr. 13, Berlin, Germany

It is known (see D.S. Mitrinović. Analytic Inequalities. Springer, 1970, item 3.6.15 on p. 272 and item 3.6.17 on p. 273) that

$$2\frac{x-1}{x+1} \leq \ln x \leq \frac{x-1}{\sqrt{x}} \quad \text{for } x \geq 1.$$

Taking $x = F_{k+1}/F_k$, $k \in N$, using $F_{k+1} - F_k = F_{k-1}$, $F_{k+1} + F_k = F_{k+2}$, and $\ln(F_{k+1}/F_k) = \ln F_{k+1} - \ln F_k$, give

$$2\frac{F_{k-1}}{F_{k+2}} \leq \ln F_{k+1} - \ln F_k \leq \frac{F_{k-1}}{\sqrt{F_k F_{k+1}}}, \quad k \in N.$$

Summing over $k = 1, 2, \cdots, n$ and noting that $\ln F_1 = 0$, one obtains the logarithmic forms of the desired inequalities.

Also solved by Paul S. Bruckman, Pavel Trojovský (2 solutions), and the proposer.

A Divisibility Issue

B-1000 Proposed by Mihály Bencze, Romania (Vol. 43, no. 2, May 2005)

Prove that $F_{nF_n^k}$ is divisible by F_n^{k+1} for all $n \geq 1$ and $k \geq 1$.

Solution by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI

In [1] the following general result is proven:

Corollary 4: For $L,\ m,\ r \geq 0$, if F_m^L divides r, then F_m^{L+1} divides F_{mr}.

With $L = k,\ m = n$, and $r = F_n^k$, in view of the Corollary above the result follows.

[1] A.T. Benjamin and J.A. Rouse. "When Does F_m^L Divide F_n? a Combinatorial Solution." The paper can be found on Arthur Benjamin's web page.

In his solution, H.-J. Seiffert refers to two references:

1. S. Rabinowitz. "Algorithmic Manipulation of Second-Order Linear Recurrences." *The Fibonacci Quarterly* **37.2** (1999): 162-77.
2. L. Somer. "Comment on B-715." *The Fibonacci Quarterly* **31.3** (1993): 279.

Also solved by Paul S. Bruckman, H. -J. Seiffert, Pavel Trojovský, and the proposer.