ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
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PROBLEMS PROPOSED IN THIS ISSUE

**H-697** Proposed by N. Gauthier, Kingston, ON

Define $K_0 = 1$ and, for a positive integer $n$, let $K_n$ represent the sum of the cubes of the first $n$ positive integers. Then define

$$
\binom{n}{k}_K \equiv \frac{K_n K_{n-1} \cdots K_{n-k+1}}{K_k K_{k-1} \cdots K_1 K_0}, \quad \text{for } 0 \leq k \leq n.
$$

a) Show that $\binom{n}{n-k}_K = \binom{n}{k}_K$.

b) Show that $\binom{n}{k}_K = m^2$, where $m = m(n,k)$ is a positive integer.

c) Find a closed form expression for $S_n = \sum_{k=0}^{\infty} m(n,k)$.

**H-698** Proposed by Hideyuki Ohtsuka, Saitama, Japan

i) Prove that

$$
\left( \sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} = F_{n-1} F_n - \frac{(-1)^n}{3} + O \left( \frac{1}{F_n^2} \right).
$$

ii) Is it true that for all nonnegative integers $m$ we have the estimate

$$
\left( \sum_{k=n}^{\infty} \frac{1}{F_k F_{k+m}} \right)^{-1} = \sum_{k=1}^{n-1} F_k F_{k+m} + \frac{1}{3} F_{m-2} (-1)^n + O \left( \frac{1}{F_n^2} \right),
$$

where the constant implied by the above $O$ might depend on $m$?
Proposed by Ovidiu Furdui, Cluj, Romania and Huizeng Qin, Shandong, China

Let \( k \geq 0 \) be a natural number and let \((x_n)_{n \in \mathbb{N}}\) be the sequence defined by

\[
x_n = n^{\frac{1}{2}} \Gamma \left(-2k + \frac{1}{2}\right) \Gamma \left(-2k + \frac{3}{2}\right) \cdots \Gamma \left(-2k + \frac{1}{n}\right) \]
\[
- n^{\frac{1}{2}} (-1)^{n-1} \Gamma \left(-2k + 1 + \frac{1}{2}\right) \Gamma \left(-2k + 1 + \frac{3}{2}\right) \cdots \Gamma \left(-2k + 1 + \frac{1}{n}\right),
\]

where \( \Gamma \) denotes the classical Gamma function. Find \( \lim_{n \to \infty} \frac{x_n}{n} \).

SOLUTIONS

Some Telescoping Series

Proposed by N. Gauthier, Kingston, ON (Vol. 46, No. 4, November 2008)

For \( x \neq 0 \) an indeterminate and for an integer \( n \geq 0 \), consider the generalized Fibonacci and Lucas polynomials \( \{f_n\}_n \) and \( \{l_n\}_n \), respectively, given by the following recurrences

\[
f_{n+2} = xf_{n+1} + f_n \quad n \geq 0, \quad \text{where} \quad f_0 = 0, \ f_1 = 1;
\]
\[
l_{n+2} = xl_{n+1} + l_n \quad n \geq 0, \quad \text{where} \quad l_0 = 2, \ l_1 = x.
\]

Find closed-form expressions for the following sums:

(a) \( \sum_{k=1}^{m} (-1)^{kn} \frac{1}{f_{(k+1)n} f_{kn}}, \quad m, \ n \geq 1; \)

(b) \( \sum_{k=0}^{m} (-1)^{kn} \frac{1}{l_{(k+1)n} l_{kn}}, \quad m, \ n \geq 0; \)

(c) \( \sum_{k=1}^{m} (-1)^{kn} \frac{f_{(2k+1)n}}{f_{(k+1)n} f_{kn}^2}, \quad m, \ n \geq 1; \)

(d) \( \sum_{k=0}^{m} (-1)^{kn} \frac{f_{(2k+1)n}}{l_{(k+1)n} l_{kn}^2}, \quad m, \ n \geq 0; \)

(e) \( \sum_{k=0}^{m} (-1)^{kn} \frac{f_{(2k+1)n} [f_{(2k+1)n} + f_{kn}^2]}{l_{(k+1)n}^4 l_{kn}^4}, \quad m, \ n \geq 0. \)

Solution by the proposer

The characteristic equations for the given recurrences are identical and have roots \( \alpha = \frac{1}{2}(x + \sqrt{x^2 + 4}) \), \( \beta = \frac{1}{2}(x - \sqrt{x^2 + 4}) \), with \( \alpha \beta = -1 \) and \( \alpha + \beta = x \). The Binet form for the terms of the generalized Fibonacci sequence is \( f_n = (\alpha^n - \beta^n)/(\alpha - \beta) \) and for the Lucas sequence is \( l_n = \alpha^n + \beta^n \). We first prove two results that will simplify the proofs.

1. For integers \( r \) and \( s \), we have \( f_{r+s} = \frac{1}{2}(f_r l_s + f_s l_r) \).
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For the proof, note that
\[
f_{r+s} = \frac{1}{\alpha - \beta} (\alpha^{r+s} - \beta^{r+s})
\]
\[
= \frac{1}{2(\alpha - \beta)} ((\alpha^{r+s} - \beta^{r+s} + \alpha \beta^s - \alpha^s \beta^r) + (\alpha^{r+s} - \beta^{r+s} - \alpha^r \beta^s + \alpha^s \beta^r))
\]
\[
= \frac{1}{2(\alpha - \beta)} ((\alpha^r - \beta^r)(\alpha^s + \beta^s) + (\alpha^s - \beta^s)(\alpha^r + \beta^r))
\]
\[
= \frac{1}{2} (f_r l_s + f_s l_r),
\]
which is what we wanted to prove.

2. For integers \( r \) and \( s \), we have that
\[
f_{r-s} = \frac{(-1)^s}{2} (f_r l_s - f_s l_r).
\]
For the proof,
\[
f_{r-s} = \frac{1}{\alpha - \beta} (\alpha^{r-s} - \beta^{r-s})
\]
\[
= \frac{1}{(\alpha - \beta)} (\alpha \beta^s (\alpha)^{-s} - \alpha^s (\alpha \beta)^{-s})
\]
\[
= \frac{(-1)^s}{2(\alpha - \beta)} ((\alpha^{r+s} - \beta^{r+s} + \alpha \beta^s - \alpha^s \beta^r) - (\alpha^{r+s} - \beta^{r+s} - \alpha^r \beta^s + \alpha^s \beta^r))
\]
\[
= \frac{(-1)^s}{2(\alpha - \beta)} ((\alpha^r - \beta^r)(\alpha^s + \beta^s) - (\alpha^s - \beta^s)(\alpha^r + \beta^r))
\]
\[
= \frac{(-1)^s}{2} (f_r l_s - f_s l_r),
\]
which is what we wanted to prove.

Now, with \( (n, k) \) integers, put \( r := n(k+1) \) and \( s := nk \) in the above formulas and rearrange the results in either one of the following forms, by dividing by \( f_{n(k+1)} f_{nk} \) or by \( l_{n(k+1)} l_{nk} \), as the case may be, to get that:

\[
(1a) \quad \frac{f_{n(2k+1)}}{f_{n(k+1)} f_{nk}} = \frac{1}{2} \left( \frac{l_{nk}}{f_{nk}} + \frac{l_{n(k+1)+1}}{f_{n(k+1)}} \right), \quad n \geq 1, \ k \geq 1;
\]
\[
(1b) \quad \frac{l_{n(2k+1)+1}}{f_{n(k+1)} l_{nk}} = \frac{1}{2} \left( \frac{f_{n(k+1)+1}}{l_{n(k+1)}} + \frac{f_{nk}}{l_{nk}} \right), \quad n \geq 0, \ k \geq 0;
\]
\[
(2a) \quad \frac{(-1)^k f_n}{f_{n(k+1)} f_{nk}} = \frac{1}{2} \left( \frac{l_{nk}}{f_{nk}} - \frac{l_{n(k+1)+1}}{f_{n(k+1)+1}} \right), \quad n \geq 1, \ k \geq 1;
\]
\[
(2b) \quad \frac{(-1)^k f_n}{l_{n(k+1)} l_{nk}} = \frac{1}{2} \left( \frac{f_{n(k+1)+1}}{l_{n(k+1)+1}} - \frac{f_{nk}}{l_{nk}} \right), \quad n \geq 0, \ k \geq 0.
\]

To find the sought closed forms, we first invoke (2a) and sum the resulting telescoping series. This gives the desired closed form for sum (a) upon division of (2a) by \( f_n \):

Closed form for (a):
\[
\sum_{k=1}^{m} \frac{(-1)^k}{f_{n(k+1)} f_{nk}} = \frac{1}{2 f_n} \sum_{k=1}^{m} \left( \frac{l_{nk}}{f_{nk}} - \frac{l_{n(k+1)+1}}{f_{n(k+1)+1}} \right) = \frac{1}{2 f_n} \left( \frac{l_n}{f_n} - \frac{l_{n(m+1)+1}}{f_{n(m+1)+1}} \right), \quad m \geq 1, \ n \geq 1.
\]

We proceed similarly for sum (b) and get, upon division of (2b) by \( f_n \), that:
Closed form for (b):
\[ \sum_{k=0}^{m} (-1)^{nk} \frac{1}{l_{n(k+1)} l_{nk}} = \frac{1}{2f_n} \sum_{k=0}^{m} \left( \frac{f_{n(k+1)}}{l_{n(k+1)}} - \frac{f_{nk}}{l_{nk}} \right) = \frac{f_{n(m+1)}}{2f_n l_{n(m+1)}}, \quad m \geq 0, \ n \geq 1. \]

To proceed further, form the product of equation (1a) by equation (2a) and get that:
\[ (3) \quad (-1)^{nk} \frac{f_n f_n f_n(2k+1)}{f_{n(k+1)}^2 f_{nk}^2} = \frac{1}{4} \left( \frac{f_{n(k+1)}^2}{f_{n(k+1)}^2} - \frac{f_{nk}^2}{f_{nk}^2} \right). \]

Dividing this result by \( f_n \) then gives the summand in (c) and the resulting sum telescopes:
Closed form for (c):
\[ \sum_{k=1}^{m} (-1)^{nk} \frac{f_{n(2k+1)}}{f_{n(k+1)}^2 f_{nk}^2} = \frac{1}{4f_n} \left( \frac{f_{n(m+1)}^2}{f_{n(m+1)}^2} - \frac{f_n^2}{f_n^2} \right), \quad m \geq 1, \ n \geq 1. \]

Similarly, form the product of (1b) by (2b) and get:
\[ (4) \quad (-1)^{nk} \frac{f_n f_n f_n(2k+1)}{l_{n(k+1)}^2 l_{nk}^2} = \frac{1}{4} \left( \frac{l_{n(k+1)}^2}{l_{n(k+1)}^2} - \frac{l_{nk}^2}{l_{nk}^2} \right). \]

This gives the summand of sum (d) upon division by \( f_n \) and the sum collapses to give:
Closed form for (d):
\[ \sum_{k=0}^{m} (-1)^{nk} \frac{f_{n(2k+1)}}{l_{n(k+1)}^2 l_{nk}^2} = \frac{f_{n(m+1)}^2}{4f_n^2 l_{n(m+1)}^2}, \quad m \geq 0, \ n \geq 1. \]

Next, take the square of equation (1b) and add the result to the square of equation (2b). This gives:
\[ (5) \quad \frac{f_{n(2k+1)}^2}{l_{n(k+1)}^2 l_{nk}^2} + \frac{f_n^2}{l_{n(k+1)}^2 l_{nk}^2} = \frac{1}{2} \left( \frac{f_{n(k+1)}^2}{l_{n(k+1)}^2} + \frac{f_{nk}^2}{l_{nk}^2} \right). \]

Multiplication of this result by (4) then gives:
\[ (6) \quad (-1)^{nk} \frac{f_n f_n f_n(2k+1)}{l_{n(k+1)}^4 l_{nk}^4} = \frac{1}{8} \left( \frac{f_{n(k+1)}^4}{l_{n(k+1)}^4} - \frac{f_{nk}^4}{l_{nk}^4} \right). \]

This gives the summand in sum (e) upon division by \( f_n \) and we get the desired result due to the collapsing of the series:
Closed form for (e):
\[ \sum_{k=0}^{m} (-1)^{nk} \frac{f_{n(2k+1)}(f_{n(k+1)}^2 + f_n^2)}{l_{n(k+1)}^4 l_{nk}^4} = \frac{f_{n(m+1)}^4}{8f_n l_{n(m+1)}^4}, \quad m \geq 0, \ n \geq 1. \]

Also solved by Paul S. Bruckman.
Integral Power Binomial Weighted Sums of Generalized Fibonacci Polynomials

H-681 Proposed by N. Gauthier, Kingston, ON
(Vol. 47, No. 1, February 2009/2010)

For a real variable $z \neq 0$ consider the sets of generalized Fibonacci and Lucas polynomials, 
\[ \{ f_n = f_n(z) : n \in \mathbb{Z} \} \quad \text{and} \quad \{ l_n = l_n(z) : n \in \mathbb{Z} \}, \]
given by the recurrences
\[ f_{n+2} = zf_{n+1} + f_n, \quad \text{and} \quad l_{n+2} = zl_{n+1} + l_n, \quad \text{for all} \quad n \in \mathbb{Z}, \]
with $f_0 = 0$, $f_1 = 1$, $l_0 = 2$, $l_1 = z$. Note that $f_{-n} = (-1)^{n+1} f_n$ and $l_{-n} = (-1)^n l_n$. Let $r$ be a nonnegative integer and $p$, $q$ be positive integers.

(a) Prove that
\[ \sum_{k \geq 0} (-1)^k \binom{r}{k} f_p^k f_{p+q}^{r-k} l_{qk} = (-1)^{q+1} r f_p^q f_{p+q}^{r-1} l_{p-r-(p+q)}. \]

(b) Find a general formula for $\sum_{k \geq 0} (-1)^k m \binom{r}{k} f_p^k f_{p+q}^{r-k} l_{qk}$ for any nonnegative integer $m$.

Solution by the proposer

The characteristic equations for the given recurrences are identical and have roots $\alpha = \frac{1}{2}(z + \sqrt{z^2 + 4})$, $\beta = \frac{1}{2}(z - \sqrt{z^2 + 4})$, with $\alpha \beta = -1$ and $\alpha + \beta = z$. The Binet form for the terms of the generalized Fibonacci sequence is $f_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ and for the Lucas sequence is $l_n = \alpha^n + \beta^n$.

To prove identity (a), we use the following lemmas.

Lemma 1. For $x$ a variable and $r$ a nonnegative integer, we have
\[ \sum_{k \geq 0} (-1)^k \binom{r}{k} (1 + x)^k = r x^{r-1}(1 + x). \]

Proof. First note that
\[ \sum_{k \geq 0} (-1)^k \binom{r}{k} (1 + x)^k = x^r, \]
which follows from the binomial expansion of $x^r = (-1 + (1 + x))^r$ in powers of $(1 + x)$. Then apply the differential operator $(1 + x) \frac{d}{dx}$ to this result and get that
\[ \sum_{k \geq 0} (-1)^k \binom{r}{k} (1 + x)^k = r x^{r-1}(1 + x); \quad r \geq 0, \]
which proves Lemma 1. \qed

Lemma 2. For positive integers $(p, q)$, the solution of the following simultaneous equations
\[ 1 + u \alpha^p = w \alpha^{-q}, \quad 1 + u \beta^p = w \beta^{-q}, \]
for the unknowns $u$ and $w$ is:
\[ u = -\frac{f_q}{f_{p+q}}, \quad w = (-1)^q \frac{f_p}{f_{p+q}}, \]
Theorem. One can get at once that \( \alpha^q + u\alpha^{p+q} = w = \beta^q + u\beta^{p+q} \). Hence, since \( p + q \neq 0 \), we get that

\[
u = \frac{\beta^q - \alpha^q}{\alpha^{p+q} - \beta^{p+q}} = -\frac{f_q}{f_{p+q}}.
\]

Similarly, one can see that \(-\alpha^{-p} + w\alpha^{-(p+q)} = u = -\beta^{-p} + w\beta^{-(p+q)} \). Hence, we get using the Binet formula for the Fibonacci polynomials that

\[
w = \frac{\alpha^{-p} - \beta^{-p}}{\alpha^{-(p+q)} - \beta^{-(p+q)}} = \frac{f_{-p}}{f_{-(p+q)}} = \frac{(-1)^{p+1}f_p}{(-1)^{p+q+1}f_{p+q}} = (-1)^q\frac{f_p}{f_{p+q}},
\]

which proves Lemma 2.

We now prove summation formula (a). To do so, first note that \( \alpha \beta = -1 \) implies that \( \alpha^{-q} = (-1)^q\beta^q \). We use Lemma 1 and 2 with

\[
x = u\alpha^p = -\frac{f_q}{f_{p+q}}\alpha^p, \quad (1 + x) = w\alpha^{-q} = (-1)^qw\beta^q = \frac{f_p}{f_{p+q}}\beta^q,
\]

to get that

\[
\sum_{k \geq 0} (-1)^{r-k}k\left(\begin{array}{c} r \\ k \end{array}\right)\left(\frac{f_p}{f_{p+q}}\beta^q\right)^k = r\left(-\frac{f_q}{f_{p+q}}\alpha^p\right)^{r-1}\left(\frac{f_p}{f_{p+q}}\beta^q\right) = (-1)^{q+r-1}\frac{f_pf_{q-1}}{f_{p+q}}\alpha^{p-r-(p+q)}.
\]

Repeating the exercise with

\[
x = -\frac{f_q}{f_{p+q}}\beta^p \quad \text{and} \quad (1 + x) = \frac{f_p}{f_{p+q}}\alpha^q,
\]

gives that

\[
\sum_{k \geq 0} (-1)^{r-k}k\left(\begin{array}{c} r \\ k \end{array}\right)\frac{f_p^k}{f_{p+q}^k}\alpha^q = (-1)^{q+r-1}\frac{f_p^k}{f_{p+q}^k}\beta^{p-r-(p+q)}.
\]

Finally, add these last two results together and multiply the resulting equation by \((-1)^rf_{p+q}^r\) to get identity (a):

\[
\sum_{k \geq 0} (-1)^{r-k}k\left(\begin{array}{c} r \\ k \end{array}\right)f_p^k f_{p+q}^{r-k} q_k = (-1)^{q+r+1}r f_p^r f_{q-1} f_{p-(q+1)}, \quad r \geq 0.
\]

To generalize the problem as requested in part (b), we will use the following lemma.

Lemma 3. For \( x \) an arbitrary variable and for an integer \( r \neq 0 \), we have

\[
\sum_{k \geq 0} (-1)^{r-k}k^m\left(\begin{array}{c} r \\ k \end{array}\right)(1 + x)^k = \sum_{n=0}^m (r)_n S_n^{(m)}x^r - n(1 + x)^n,
\]

where \( \{S_n^{(m)} : 0 \leq m, 0 \leq n \leq m\} \) is the augmented set of Stirling numbers of the second kind, including the \( n = 0 \) elements, \( S_0^{(m)} = \delta_{m,0} \). Also, by definition, for \( n \geq 1 \), \( (r)_n = r(r - 1)\cdots(r - n + 1) \) and for \( n = 0 \), \( (r)_0 = 1 \).

Proof. For \( m \geq 0 \), consider the differential operator \( (1 + x)\frac{d}{dx} \) and apply it to the formula

\[
\sum_{k \geq 0} (-1)^{r-k}\left(\begin{array}{c} r \\ k \end{array}\right)(1 + x)^k = x^r
\]

where
To prove the above claim, note that it is true for $m = 0$. After noting that $(1 + x)^m (1 + x)^k = k^m (1 + x)^k$ as well as the fact that \( (1 + x) \frac{dx}{dx} \) generates an $m + 1$-term expansion in $\{x^{r-n}(1+x)^n : 0 \leq n \leq m\}$, we claim that the following holds for nonnegative $r, m$:

\[
\sum_{k \geq 0} (-1)^{r-k} k^m \binom{r}{k} (1 + x)^k = \sum_{n=0}^{m} (r)_n a_n^{(m)} x^{r-n} (1 + x)^n.
\]

The unknown coefficients, $\{a_n^{(m)} : 0 \leq m, 0 \leq n \leq m\}$, are to be determined by solving the following linear recurrence:

\[
a_n^{(m+1)} = n a_n^{(m)} + a_{n-1}^{(m)}, \quad a_0^{(0)} = 1, \quad a_{-1}^{(m)} = a_{m+1}^{(m)} = 0.
\]

To prove the above claim, note that it is true for $m = 0$ if we convene that $k^0 = 1$ for all $k \geq 0$. So, assuming that the above formula is true for $m$, consider

\[
(1 + x) \frac{dx}{dx} = (1 + x)^{m+1} \frac{dx}{dx} \left( (1 + x) \frac{dx}{dx} \right)^m x^r = (1 + x)^m \frac{dx}{dx} \left( (1 + x) \frac{dx}{dx} \right)^m x^r.
\]

Upon invoking the above expressions for $(1 + x)^{m+1} x^r$ and of $(1 + x) x^r$ in powers of $(1 + x)/x$, we get that

\[
\sum_{n=0}^{m+1} (r)_n a_n^{(m+1)} x^{r-n} (1 + x)^n = (1 + x) \frac{dx}{dx} \sum_{n=0}^{m} (r)_n a_n^{(m)} x^{r-n} (1 + x)^n
\]

\[
= \sum_{n=0}^{m} (r)_n a_n^{(m)} \left((r - n)x^{r-n-1}(1 + x)^{n+1} + nx^{r-n}(1 + x)^n\right)
\]

\[
= \sum_{n=0}^{m+1} (r)_n a_n^{(m+1)} x^{r-(n+1)} (1 + x)^{n+1} + \sum_{n=0}^{m} n (r)_n a_n^{(m)} x^{r-n} (1 + x)^n
\]

\[
= \sum_{n=0}^{m+1} (r)_n \left(a_n^{(m+1)} + n a_n^{(m)}\right) x^{r-n} (1 + x)^n.
\]

To go from the penultimate line to the last one above, we shifted the summation index in the first sum by one unit. Then we defined $a_{-1}^{(m)} = 0, a_{m+1}^{(m)} = 0$ and extended the limits of both sums from 0 to $m+1$. This result then gives the recurrence for the unknown coefficients, which is the recurrence for the augmented Stirling numbers of the second kind, $S_n^{(m)}$. We therefore conclude that $\{a_n^{(m)} = S_n^{(m)} : 0 \leq m, 0 \leq n \leq m\}$ and Lemma 3 is proved.

Now, to obtain the generalization requested in (b) of the problem statement, we invoke Lemmas 2 and 3 and proceed as we did to prove identity (a). We then get the following two
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equations:

\[
\sum_{k \geq 0} (-1)^{r-k} k^m \binom{r}{k} \frac{f_p^k}{f_{p+q}^k} \beta^{q_k} = \sum_{n=0}^{m} (r)_n S_n^{(m)} \left( \frac{-f_p}{f_{p+q}} \right)^{r-n} \left( \frac{f_p}{f_{p+q}} \right)^n \\
= \sum_{n=0}^{m} (-1)^{(q+1)n+r} (r)_n S_n^{(m)} \frac{f_p^{r-n} f_q}{f_{p+q}^{r+q}} \alpha^{r-(p+q)n},
\]

\[
\sum_{k \geq 0} (-1)^{r-k} k^m \binom{r}{k} \frac{f_p^k}{f_{p+q}^k} \alpha^{q_k} = \sum_{n=0}^{m} (-1)^{(q+1)n+r} (r)_n S_n^{(m)} \frac{f_p^{r-n} f_q}{f_{p+q}^{r+q}} \beta^{r-(p+q)n}.
\]

Adding together these two equations and multiplying the result by \((-1)^r f_p^r f_q^{r+q}\) then gives the sought generalization

\[
\sum_{k \geq 0} (-1)^k k^m \binom{r}{k} f_p^k f_{p+q}^{r-k} \epsilon_{r} = \sum_{n=0}^{m} (-1)^{(q+1)n+r} (r)_n S_n^{(m)} f_p^n f_q^{r-n} \epsilon_{r-(p+q)n}.
\]

This result agrees with the identity in (a) when \(m = 1\) since \(S_0^{(1)} = 0, S_1^{(1)} = 1\) and \((r)_1 = r\).

Also solved by Paul S. Bruckman and Kenneth Davenport.