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PROBLEMS PROPOSED IN THIS ISSUE

H-712 Proposed by N. Gauthier, The Royal Military College of Canada, Kingston, ON

The \( n \)th central binomial coefficient is, for an integer \( n \geq 0 \): \( B_n = \binom{2n}{n} \). Then, for a nonnegative integer \( m \), define the convolution

\[
b_m(n) = \sum_{k=0}^{n} k^m B_{n-k} B_k,
\]

where \( b_0(n) = \sum_{k=0}^{n} B_{n-k} B_k \). Prove the following recurrence,

\[
b_m(n) = \frac{2^{2n-m}(2m-1)!!(n)_m}{m!} - \sum_{k=1}^{m-1} S_m^{(k)} b_k(n).
\]

In this expression, the sum in the right–hand side is taken to vanish when \( m = 0, 1 \), and the coefficients are Stirling numbers of the first kind, \( \{S_m^{(k)} : 1 \leq k \leq m\} \). Also,

\[
(2m-1)!! = 1 \cdot 3 \cdot 5 \cdots (2m-1); \quad (n)_m = n(n-1)\cdots(n-m+1),
\]

where, by convention, \( (2m-1)!! = 1 \) and \( (n)_m = 1 \) for \( m = 0 \).

H-713 Proposed by Hideyuki Ohtsuka, Saitama, Japan

Determine

\[
(1) \quad \sum_{k=1}^{\infty} \frac{2^k F_{2^k}}{L_{3.2^k}} \quad \text{and} \quad (2) \quad \sum_{k=1}^{\infty} \frac{2^k F_{3^k}}{L_{2.2^k} L_{3.2^k}}.
\]

H-714 Proposed by N. Gauthier, The Royal Military College of Canada, Kingston, ON

Let \( n \) be a positive integer. Find a closed–form expression for the following sum:

\[
S(n) = \sum_{k=1}^{n} k^2 \binom{2n-2k}{n-k} \binom{2k}{k}.
\]
The Tribonacci numbers \( T_n \) satisfy
\[
T_0 = 0, \quad T_1 = T_2 = 1, \quad T_{n+3} = T_{n+2} + T_{n+1} + T_n \quad \text{for} \quad n \geq 0.
\]

Find explicit formulas for

(1) \[ \sum_{k=1}^{n} T_k^2 \]

and

(2) \[ \sum_{k=1}^{n} (T_k^2 - T_{k+1}T_{k-1})^2. \]

**SOLUTIONS**

Catalan’s Constant, \( \pi \) and \( \ln 2 \)

**H-691** Proposed by Ovidiu Furdui, Cluj, Romania and Huizeng Qin, Shandong, China

(Vol. 47, No. 3, August 2009/2010)

Find the value of
\[
\sum_{n=1}^{\infty} (-1)^n \left( \ln 2 - \frac{1}{n+1} - \frac{1}{n+2} - \cdots - \frac{1}{2n} \right)^2.
\]

Solution by Khristo N. Boyadzhiev, Ohio Northern University, Ohio

Let \( \sigma \) be the sum to be evaluated. We shall see that
\[
\sigma = \frac{G}{2} + \frac{13\pi^2}{192} - \frac{7(\ln 2)^2}{8} - \frac{\pi \ln 2}{8}, \quad (1)
\]
where \( G \) is the Catalan constant to be defined later.

First we use a well-known identity (see [3])
\[
\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = \sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k}.
\]

At the same time,
\[
\ln 2 = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}.
\]

Thus,
\[
\ln 2 - \frac{1}{n+1} - \frac{1}{n+2} - \cdots - \frac{1}{2n} = \sum_{k=2n+1}^{\infty} \frac{(-1)^{k-1}}{k} = \int_0^1 x^{2n} \frac{dx}{1+x}.
\]
The last equality is easy to establish by expanding $1/(1 + x)$ in power series and integrating termwise. Next we write

$$\sigma = \sum_{n=1}^{\infty} (-1)^n \left( \int_0^1 \frac{x^{2n} dx}{1 + x} \right)^2$$

$$= \sum_{n=1}^{\infty} (-1)^n \left( \int_0^1 \frac{x^{2n} dx}{1 + x} \right) \left( \int_0^1 \frac{y^{2n} dy}{1 + y} \right)$$

$$= \sum_{n=1}^{\infty} (-1)^n \int_0^1 \int_0^1 \frac{x^{2n} y^{2n} dx dy}{(1 + x)(1 + y)}$$

$$= \int_0^1 \int_0^1 \left( \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n} y^{2n}}{(1 + x)(1 + y)} \right) dx dy$$

$$= - \int_0^1 \int_0^1 \frac{x^2 y^2 dx dy}{(1 + x^2 y^2)(1 + x)(1 + y)}.$$  

Here, we set $y = u/x$ to get

$$-\sigma = \int_0^1 \left( \int_0^x \frac{u^2 du}{(1 + u^2)(u + x)} \right) \frac{dx}{1 + x}$$

$$= \int_0^1 \left( \frac{x^2 \ln 2}{1 + x^2} + \frac{\ln(1 + x^2) - x \arctan x}{2(1 + x^2)} - \frac{1 + x^2}{1 + x} \right) \frac{dx}{1 + x}$$

$$= \ln 2 \int_0^1 \frac{x^2 dx}{(1 + x^2)(1 + x)} + \frac{1}{2} \int_0^1 \frac{\ln(1 + x^2) dx}{(1 + x^2)(1 + x)} + \int_0^1 \frac{-x \arctan x dx}{(1 + x^2)(1 + x)}; \quad (2)$$

i.e.,

$$-\sigma = A \ln 2 + \frac{1}{2} B + C,$$  

where $A, B, C$ are the corresponding integrals in (2). We calculate them one by one. The first one is very easy:

$$A = \frac{3 \ln 2}{4} - \frac{\pi}{8}.$$ 

Next,

$$B = \frac{1}{2} \left( \int_0^1 \frac{\ln(1 + x^2) dx}{1 + x} + \int_0^1 \frac{\ln(1 + x^2) dx}{1 + x^2} - \int_0^1 x \ln(1 + x^2) dx \right).$$

We have

$$\int_0^1 \frac{x \ln(1 + x^2) dx}{1 + x^2} = \frac{1}{2} \int_0^1 \ln(1 + x^2) d \ln(1 + x^2) = \frac{(\ln 2)^2}{4},$$

$$\int_0^1 \frac{\ln(1 + x^2) dx}{1 + x} = \frac{\pi \ln 2}{2} - G$$  

(from tables, $G$ is the Catalan constant; see, for example, 4.296.5 in [2]),

$$\int_0^1 \frac{\ln(1 + x^2) dx}{1 + x} = \frac{3(\ln 2)^2}{4} - \frac{\pi^2}{48}$$

(computed by hand, solution available). Therefore,

$$B = \frac{1}{2} \left( \frac{(\ln 2)^2}{2} - \frac{\pi^2}{48} + \frac{\pi \ln 2}{2} - G \right).$$
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Finally,
\[
\int_0^1 -x \arctan x \, dx \frac{1}{(1 + x^2)(1 + x)} = \frac{1}{2} \int_0^1 \frac{\arctan x \, dx}{1 + x} - \frac{1}{2} \int_0^1 \frac{x \arctan x \, dx}{1 + x^2} - \frac{1}{2} \int_0^1 \frac{\arctan x \, dx}{1 + x^2},
\]
where
\[
\int_0^1 \frac{\arctan x \, dx}{1 + x} = \frac{\pi \ln 2}{8}
\]
(evaluated in Problem 833 in [1]; also in [2], 4.535.1).

\[
\int_0^1 \frac{\arctan x \, dx}{1 + x^2} = \frac{1}{2} (\arctan x)^2 \bigg|_0^1 = \frac{\pi^2}{8},
\]
\[
\int_0^1 \frac{x \arctan x \, dx}{1 + x^2} = \frac{\pi \ln 2}{8} - \frac{1}{2} \int_0^1 \frac{\ln(1 + x^2) \, dx}{1 + x^2}
\]
\[
= \frac{\pi \ln 2}{8} - \frac{1}{2} \left( \frac{\pi \ln 2}{2} - G \right)
\]
\[
= \frac{G}{2} - \frac{\pi \ln 2}{8}
\]
(after integration by parts and using (4); the integral can also be reduced to 4.531.7 in [2]).

Thus,
\[
C = \frac{1}{2} \left( \frac{\pi \ln 2}{4} - \frac{\pi^2}{8} - \frac{G}{2} \right).
\]
From (3), we obtain (1).

REFERENCES


Also solved by Kenneth B. Davenport and the proposers.

Closed Forms For Trigonometric Sums


Let \( q \geq 1, \ N \geq 3 \) be integers and define \( Q = \left\lfloor (N - 1)/2 \right\rfloor \). Find closed form expressions for the following sums:

a) \( P_0(\theta, q) = \sum_{k=1}^{q} \frac{\sin(2k - 1)\theta}{\cos^2 k\theta \cos^2 (k - 1)\theta}; \)

b) \( R_0(\theta, q) = \sum_{k=1}^{q} \frac{\sin(2k - 1)\theta |\sin^2 \theta + \sin^2 (2k - 1)\theta|}{\cos^4 k\theta \cos^4 (k - 1)\theta}; \)

c) \( P_1(N) = \sum_{k=1}^{Q} \frac{k \sin \frac{(2k-1)\pi}{N}}{\cos^2 k\pi \cos^2 (k-1)\pi}; \)

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d) \( R_1(N) = \sum_{k=1}^{Q} k \sin \left( \frac{(2k-1)\pi}{N} \right) \left[ \frac{\sin^2 \frac{\pi}{N} + \sin^2 \left( \frac{(2k-1)\pi}{N} \right)}{\cos^4 \frac{k\pi}{N} \cos^4 \left( \frac{(k-1)\pi}{N} \right)} \right] \).

**Solution by the proposer**

To obtain the sought closed–form expressions, we first prove three lemmas.

**Lemma 1.** For \( k \) a positive integer and \( \theta \) a real variable such that \( 0 < k\theta < \pi/2 \), the following relation holds:
\[
\frac{\sin \theta \sin(2k-1)\theta}{\cos^2 k\theta \cos^2 (k-1)\theta} = \tan^2 k\theta - \tan^2 (k-1)\theta.
\] (5)

**Proof.** Consider the following trigonometric identities
\[
\sin k\theta \cos (k-1)\theta - \cos k\theta \sin (k-1)\theta = \sin \theta,
\]
\[
\sin k\theta \cos (k-1)\theta + \cos k\theta \sin (k-1)\theta = \sin(2k-1)\theta,
\]
and divide the results by \( \cos k\theta \cos (k-1)\theta \). This gives
\[
\frac{\sin \theta}{\cos k\theta \cos (k-1)\theta} = \tan k\theta - \tan (k-1)\theta,
\] (6)
\[
\frac{\sin(2k-1)\theta}{\cos k\theta \cos (k-1)\theta} = \tan k\theta + \tan (k-1)\theta.
\]

Multiplying the above two relations (6), we get (5). \( \square \)

**Lemma 2.** For \( k \) a positive integer and \( \theta \) a real variable such that \( 0 < k\theta < \pi/2 \), the following holds
\[
\frac{\sin(2k-1)\theta}{\cos^4 k\theta \cos^4 (k-1)\theta} = 2 \csc \theta (\tan^4 k\theta - \tan^4 (k-1)\theta).
\] (7)

**Proof.** Square the first relation (6) and get
\[
\frac{\sin^2 \theta}{\cos^2 k\theta \cos^2 (k-1)\theta} = \tan^2 \theta + \tan^2 (k-1)\theta - 2 \tan k\theta \tan (k-1)\theta.
\] (8)

Next, we square the second relation (6) and get
\[
\frac{\sin^2(2k-1)\theta}{\cos^2 k\theta \cos^2 (k-1)\theta} = \tan^2 \theta + \tan^2 (k-1)\theta + 2 \tan k\theta \tan (k-1)\theta.
\] (9)

Then add (8) and (9) to get
\[
\frac{\sin^2 \theta + \sin^2(2k-1)\theta}{\cos^2 k\theta \cos^2 (k-1)\theta} = 2(\tan^2 k\theta + \tan^2 (k-1)\theta).
\] (10)

Multiplication of (10) by (5) and division of the result by \( \sin \theta \) then gives (7). \( \square \)

**Lemma 3.** Consider an arbitrary, well-defined sequence of functions or numbers,
\[
\{w_k : k = 1, 2, 3, \ldots\}
\]
and, for any positive integer \( q \), define the following sums
\[
s_0(q) = \sum_{k=1}^{q} w_k; \quad s_1(q) = \sum_{k=1}^{q} kw_k.
\]
Then the following holds:

$$s_1(q) = (q + 1)s_0(q) - \sum_{k=1}^{q} s_0(k).$$  \hspace{1cm} (11)

Proof. Consider the following set of \( q \) equations:

\[
\begin{align*}
w_1 + w_2 + w_3 + \cdots + w_q &= s_0(q); \\
w_2 + w_3 + \cdots + w_q &= s_0(q) - s_0(1); \\
w_3 + \cdots + w_q &= s_0(q) - s_0(2); \\
& \quad \vdots \\
w_q &= s_0(q) - s_0(q - 1).
\end{align*}
\]

Then sum the terms in the left–hand side and equate the result to the sum of the terms in the right–hand side. This gives

\[
w_1 + 2w_2 + \cdots + qw_q = qs_0(q) - \sum_{k=1}^{q-1} s_0(k).
\]

Equation (11) follows upon adding and subtracting \( s_0(q) \) in the right–hand side of this result. \( \square \)

To get a closed form for a), divide (5) by \( \sin \theta \) and note that the sum over \( 1 \leq k \leq q \) collapses. This gives:

\[
P_0(q) = \sum_{k=1}^{q} \csc \theta (\tan^2 k\theta - \tan^2 (k-1)\theta) = \csc \theta \tan^2 q\theta.
\]  \hspace{1cm} (12)

Similarly, use (7) to get a closed form for b) and get:

\[
R_0(q) = \sum_{k=1}^{q} 2 \csc \theta (\tan^4 k\theta - \tan^4 (k-1)\theta) = 2 \csc \theta \tan^4 q\theta.
\]  \hspace{1cm} (13)

We now find closed forms for \( P_1(N) \) and \( R_1(N) \), as defined in parts c) and d) of the problem statement.

To proceed, let \( q \) be an integer such that \( 1 \leq k \leq q \), with \( 0 < k\theta < \pi/2 \), and consider the two functions

\[
P_1(\theta; q) = \sum_{k=1}^{q} \frac{k \sin(2k-1)\theta}{\cos^2 k\theta \cos^2 (k-1)\theta} = (q + 1)P_0(\theta, q) - \sum_{k=1}^{q} P_0(\theta, k);
\]

\[
R_1(\theta; q) = \sum_{k=1}^{q} \frac{k \sin(2k-1)\theta (\sin^2 \theta + \sin^2 (2k-1)\theta)}{\cos^2 k\theta \cos^2 (k-1)\theta} = (q + 1)R_0(\theta, q) - \sum_{k=1}^{q} R_0(\theta, k).
\]  \hspace{1cm} (14)

The right–hand sides of the two relations (14) follow from (11) applied to the pairs \( \{P_1(\theta; q), P_0(\theta; q)\} \) and \( \{R_1(\theta; q), R_0(\theta; q)\} \) with

\[
\left\{ w_k = \frac{\sin(2k-1)\theta}{\cos^2 k\theta \cos^2 (k-1)\theta} \right\} \quad \text{and} \quad \left\{ w_k = \frac{\sin(2k-1)\theta (\sin^2 \theta + \sin^2 (2k-1)\theta)}{\cos^4 k\theta \cos^4 (k-1)\theta} \right\},
\]
respectively. With \( Q = \lfloor (N - 1)/2 \rfloor \), as in the problem statement, sums c) and d) are given by

\[
P_1(N) = P_1(\theta; q)\bigg|_{\theta = \pi/N, q = Q} \quad \text{and} \quad R_1(N) = R_1(\theta; q)\bigg|_{\theta = \pi/N, q = Q}.
\]

Using (12) and (13) into (14) then gives, with the help of (15):

\[
P_1(N) = \csc \frac{\pi}{N} \left( (Q + 1) \tan^2 \frac{Q\pi}{N} - \sum_{k=1}^{Q} \tan^2 k\frac{\pi}{N} \right);
\]

\[
R_1(N) = 2 \csc \frac{\pi}{N} \left( (Q + 1) \tan^4 \frac{Q\pi}{N} - \sum_{k=1}^{Q} \tan^4 k\frac{\pi}{N} \right).
\]

We finally find \( \sum_{k=1}^{Q} \tan^{2m} \frac{k\pi}{N} \) for \( m = 1, 2 \), by invoking the general results obtained in [1]. According to equation (31) of [1],

\[
\sum_{k=1}^{Q} \tan^{2m} \frac{k\pi}{N} = \sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} S_{2r}(N), \quad m = 1, 2, 3, \ldots
\]

where

\[
S_{2r}(N) = \sum_{k=1}^{Q} \sec^{2r} \frac{k\pi}{N}, \quad r \geq 0.
\]

For \( r = 0 \), we have

\[
S_0(N) = Q,
\]

and for \( r = 1 \) we have, from (26) and (27) of [1]:

\[
S_r(N) = \begin{cases} 
\sum_{k=1}^{r} \frac{a_{k,r}(N^{2k} - 2^{2k})}{(2^{2k} - 1)} & r \geq 1 \quad N \text{ even}, \\
\sum_{k=1}^{r} \frac{a_{k,r}(N^{2k} - 1)}{2^{2k}} & r \geq 1 \quad N \text{ odd}.
\end{cases}
\]

The \( a_{k,r} \) coefficients that appear in these expressions are calculated as shown in [1]. For the cases of interest here, we need \( a_{1,1} = \frac{1}{6}, a_{1,2} = \frac{1}{9}, \) and \( a_{2,2} = \frac{1}{90} \) (see second Table, p. 271 of [1]). These values give:

**For \( N \) even,**

\[
S_0(N) = \frac{N - 2}{2}, \quad S_2(N) = a_{1,1}(N^2 - 4) = \frac{(N - 2)(N + 2)}{6},
\]

\[
S_4(N) = a_{1,2}(N^2 - 4) + a_{2,2}(N^4 - 16) = \frac{(N - 2)(N + 2)(N^2 + 14)}{90}.
\]

**For \( N \) odd,**

\[
S_0(N) = \frac{N - 1}{2}, \quad S_2(N) = 3a_{1,1}(N^2 - 1) = \frac{(N - 1)(N + 1)}{2},
\]

\[
S_4(N) = 3a_{1,2}(N^2 - 1) + 15a_{2,2}(N^4 - 1) = \frac{(N - 1)(N + 1)(N^2 + 3)}{6}.
\]
Terms are arranged to highlight common factors. Next collect terms and factorize to get
\[
\sum_{k=1}^{Q} \tan^2 \frac{k\pi}{N} = S_2(N) - S_0(N) = \begin{cases} 
\frac{(N-1)(N-2)}{N(N-1)} & N \text{ even}, \\
\frac{6}{N(N-1)} & N \text{ odd}, 
\end{cases}
\]
\[
\sum_{k=1}^{Q} \tan^4 \frac{k\pi}{N} = S_4(N) - 2S_2(N) + S_0(N) = \begin{cases} 
\frac{(N-1)(N-2)(N^2+3N-13)}{N(N-1)(N^2+N-3)} & N \text{ even}, \\
\frac{90}{N(N-1)(N^2+N-3)} & N \text{ odd}. 
\end{cases}
\]
These results can now be inserted in (16) to provide the sought closed forms and we find:
\[
P_1(N) = \begin{cases} 
csc \frac{\pi}{N} \left( \frac{N-2}{2N} \tan^2 \frac{N-1}{2} \pi - \frac{N-1}{6} \right) & N \text{ even}, \\
csc \frac{\pi}{N} \left( \frac{N+1}{2} \tan^2 \frac{N-1}{2} \pi - \frac{N(N-1)}{2} \right) & N \text{ odd}, 
\end{cases}
\]
\[
R_1(N) = \begin{cases} 
2 \csc \frac{\pi}{N} \left( \frac{N-2}{2N} \tan^4 \frac{N-1}{2} \pi - \frac{N-1}{90} \right) & N \text{ even}, \\
2 \csc \frac{\pi}{N} \left( \frac{N+1}{2} \tan^4 \frac{N-1}{2} \pi - \frac{N(N-1)(N^2+3N-13)}{6} \right) & N \text{ odd}. 
\end{cases}
\]
This completes the proof of the problem.

References

Also solved by Paul S. Bruckman.

Errata: In the solution to H-690, the expression
\[
(-1)^{k(m+1)} \sum_{k=1}^{n} \left\{ F_k^{2m} L_m + \sum_{i=1}^{[m/2]} \sum_{r=1}^{i} \frac{m}{i} \binom{m-i-1}{i-1} \binom{i}{r} (-1)^{kr} F_k^{2(m-r)} \right\}
\]
should be
\[
\sum_{k=1}^{n} \left\{ (-1)^{k(m+1)} F_k^{2m} L_m + \sum_{i=1}^{[m/2]} \sum_{r=1}^{i} \frac{m}{i} \binom{m-i-1}{i-1} \binom{i}{r} (-1)^{k(m+r+1)} F_k^{2(m-r)} \right\}.
\]