ADVANCED PROBLEMS AND SOLUTIONS

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PROBLEMS PROPOSED IN THIS ISSUE

H-712 Proposed by N. Gauthier, The Royal Military College of Canada, Kingston, ON

The nth central binomial coefficient is, for an integer $n \geq 0$: $B_n = \binom{2n}{n}$. Then, for a nonnegative integer m, define the convolution

$$b_m(n) = \sum_{k=0}^{n} k^m B_{n-k} B_k,$$

where $b_0(n) = \sum_{k=0}^{n} B_{n-k} B_k$. Prove the following recurrence,

$$b_m(n) = \frac{2^{2n-m}(2m-1)!!(n)_m}{m!} - \sum_{k=1}^{m-1} S_m^{(k)} b_k(n).$$

In this expression, the sum in the right–hand side is taken to vanish when $m = 0, 1$, and the coefficients are Stirling numbers of the first kind, $\{S_m^{(k)} : 1 \leq k \leq m\}$. Also,

$$(2m-1)!! = 1 \cdot 3 \cdot 5 \cdots (2m-1); \qquad (n)_m = n(n-1)\ldots(n-m+1),$$

where, by convention, $(2m-1)!! = 1$ and $(n)_m = 1$ for $m = 0$.

H-713 Proposed by Hideyuki Ohtsuka, Saitama, Japan

Determine

$$(1) \quad \sum_{k=1}^{\infty} \frac{2^k F_{2^k}}{L_{3\cdot 2^k}} \qquad \text{and} \qquad (2) \quad \sum_{k=1}^{\infty} \frac{2^k F_{2^k}^3}{L_{2\cdot 2^k} L_{3\cdot 2^k}}.$$

H-714 Proposed by N. Gauthier, The Royal Military College of Canada, Kingston, ON

Let n be a positive integer. Find a closed–form expression for the following sum:

$$S(n) = \sum_{k=1}^{n} k^2 \binom{2n-2k}{n-k} \binom{2k}{k}.$$

H-715 **Proposed by Hideyuki Ohtsuka, Saitama, Japan**

The Tribonacci numbers T_n satisfy

$$T_0 = 0,\ T_1 = T_2 = 1, \qquad T_{n+3} = T_{n+2} + T_{n+1} + T_n \qquad \text{for} \qquad n \geq 0.$$

Find explicit formulas for

$$(1) \qquad \sum_{k=1}^{n} T_k^2 \qquad \text{and} \qquad (2) \qquad \sum_{k=1}^{n} (T_k^2 - T_{k+1}T_{k-1})^2.$$

SOLUTIONS

Catalan's Constant, π and $\ln 2$

H-691 **Proposed by Ovidiu Furdui, Cluj, Romania and Huizeng Qin, Shandong, China**
(Vol. 47, No. 3, August 2009/2010)

Find the value of

$$\sum_{n=1}^{\infty} (-1)^n \left(\ln 2 - \frac{1}{n+1} - \frac{1}{n+2} - \cdots - \frac{1}{2n} \right)^2.$$

Solution by Khristo N. Boyadzhiev, Ohio Northern University, Ohio

Let σ be the sum to be evaluated. We shall see that

$$\sigma = \frac{G}{2} + \frac{13\pi^2}{192} - \frac{7(\ln 2)^2}{8} - \frac{\pi \ln 2}{8}, \tag{1}$$

where G is the Catalan constant to be defined later.

First we use a well-known identity (see [3])

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} = \sum_{k=1}^{2n} \frac{(-1)^{k-1}}{k}.$$

At the same time,

$$\ln 2 = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}.$$

Thus,

$$\ln 2 - \frac{1}{n+1} - \frac{1}{n+2} - \cdots - \frac{1}{2n} = \sum_{k=2n+1}^{\infty} \frac{(-1)^{k-1}}{k} = \int_0^1 \frac{x^{2n}dx}{1+x}.$$

The last equality is easy to establish by expanding $1/(1+x)$ in power series and integrating termwise. Next we write

$$\begin{aligned}\sigma &= \sum_{n=1}^{\infty}(-1)^n\left(\int_0^1 \frac{x^{2n}dx}{1+x}\right)^2\\ &= \sum_{n=1}^{\infty}(-1)^n\left(\int_0^1 \frac{x^{2n}dx}{1+x}\right)\left(\int_0^1 \frac{y^{2n}dy}{1+y}\right)\\ &= \sum_{n=1}^{\infty}(-1)^n\int_0^1\int_0^1 \frac{x^{2n}y^{2n}dxdy}{(1+x)(1+y)}\\ &= \int_0^1\int_0^1\left(\sum_{n=1}^{\infty}(-x^2y^2)^n\right)\frac{dxdy}{(1+x)(1+y)}\\ &= -\int_0^1\int_0^1 \frac{x^2y^2dxdy}{(1+x^2y^2)(1+x)(1+y)}.\end{aligned}$$

Here, we set $y = u/x$ to get

$$\begin{aligned}-\sigma &= \int_0^1\left(\int_0^x \frac{u^2du}{(1+u^2)(u+x)}\right)\frac{dx}{(1+x)}\\ &= \int_0^1\left(\frac{x^2\ln 2}{1+x^2}+\frac{\ln(1+x^2)}{2(1+x^2)}-\frac{x\arctan x}{1+x^2}\right)\frac{dx}{(1+x)}\\ &= \ln 2\int_0^1 \frac{x^2dx}{(1+x^2)(1+x)}+\frac{1}{2}\int_0^1\frac{\ln(1+x^2)dx}{(1+x^2)(1+x)}+\int_0^1\frac{-x\arctan x dx}{(1+x^2)(1+x)};\end{aligned} \tag{2}$$

i.e.,

$$-\sigma = A\ln 2+\frac{1}{2}B+C, \tag{3}$$

where A, B, C are the corresponding integrals in (2). We calculate them one by one. The first one is very easy:

$$A = \frac{3\ln 2}{4}-\frac{\pi}{8}.$$

Next,

$$B = \frac{1}{2}\left(\int_0^1\frac{\ln(1+x^2)dx}{1+x}+\int_0^1\frac{\ln(1+x^2)dx}{1+x^2}-\int_0^1\frac{x\ln(1+x^2)dx}{1+x^2}\right).$$

We have

$$\int_0^1\frac{x\ln(1+x^2)dx}{1+x^2} = \frac{1}{2}\int_0^1\ln(1+x^2)d\ln(1+x^2) = \frac{(\ln 2)^2}{4},$$

$$\int_0^1\frac{\ln(1+x^2)dx}{1+x} = \frac{\pi\ln 2}{2}-G \tag{4}$$

(from tables, G is the Catalan constant; see, for example, 4.296.5 in [2]),

$$\int_0^1\frac{\ln(1+x^2)dx}{1+x} = \frac{3(\ln 2)^2}{4}-\frac{\pi^2}{48}$$

(computed by hand, solution available). Therefore,

$$B = \frac{1}{2}\left(\frac{(\ln 2)^2}{2}-\frac{\pi^2}{48}+\frac{\pi\ln 2}{2}-G\right).$$

Finally,

$$\int_0^1 \frac{-x \arctan x dx}{(1+x^2)(1+x)} = \frac{1}{2}\int_0^1 \frac{\arctan x dx}{1+x} - \frac{1}{2}\int_0^1 \frac{x \arctan x dx}{1+x^2} - \frac{1}{2}\int_0^1 \frac{\arctan x dx}{1+x^2},$$

where

$$\int_0^1 \frac{\arctan x dx}{1+x} = \frac{\pi \ln 2}{8}$$

(evaluated in Problem 833 in [1]; also in [2], 4.535.1).

$$\int_0^1 \frac{\arctan x dx}{1+x^2} = \frac{1}{2}(\arctan x)^2\Big|_0^1 = \frac{\pi^2}{8},$$

$$\begin{aligned}\int_0^1 \frac{x \arctan x dx}{1+x^2} &= \frac{\pi \ln 2}{8} - \frac{1}{2}\int_0^1 \frac{\ln(1+x^2)dx}{1+x^2} \\ &= \frac{\pi \ln 2}{8} - \frac{1}{2}\left(\frac{\pi \ln 2}{2} - G\right) \\ &= \frac{G}{2} - \frac{\pi \ln 2}{8}\end{aligned}$$

(after integration by parts and using (4); the integral can also be reduced to 4.531.7 in [2]). Thus,

$$C = \frac{1}{2}\left(\frac{\pi \ln 2}{4} - \frac{\pi^2}{8} - \frac{G}{2}\right).$$

From (3), we obtain (1).

References

[1] K. Boyadzhiev and L. Glasser, *Solution to problem 833*, College Math. J., **40.4** (2009), 297–298.

[2] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, 1965.

[3] S. Wolfram, The Harmonic Number Page of the Wolfram Mathworld, `http://mathworld.wolfram.com/HarmonicNumber.html`.

Also solved by Kenneth B. Davenport and the proposers.

Closed Forms For Trigonometric Sums

H-692 Proposed by Napoleon Gauthier, Kingston, ON (Vol. 47, No. 3, August 2009/2010)

Let $q \geq 1,\ N \geq 3$ be integers and define $Q = \lfloor (N-1)/2 \rfloor$. Find closed form expressions for the following sums:

a) $P_0(\theta, q) = \displaystyle\sum_{k=1}^{q} \frac{\sin(2k-1)\theta}{\cos^2 k\theta \cos^2(k-1)\theta}$;

b) $R_0(\theta, q) = \displaystyle\sum_{k=1}^{q} \frac{\sin(2k-1)\theta[\sin^2\theta + \sin^2(2k-1)\theta]}{\cos^4 k\theta \cos^4(k-1)\theta}$;

c) $P_1(N) = \displaystyle\sum_{k=1}^{Q} \frac{k \sin\frac{(2k-1)\pi}{N}}{\cos^2\frac{k\pi}{N}\cos^2\frac{(k-1)\pi}{N}}$;

d) $R_1(N) = \sum_{k=1}^{Q} \frac{k \sin \frac{(2k-1)\pi}{N} \left[\sin^2 \frac{\pi}{N} + \sin^2 \frac{(2k-1)\pi}{N}\right]}{\cos^4 \frac{k\pi}{N} \cos^4 \frac{(k-1)\pi}{N}}.$

Solution by the proposer

To obtain the sought closed–form expressions, we first prove three lemmas.

Lemma 1. *For k a positive integer and θ a real variable such that $0 < k\theta < \pi/2$, the following relation holds:*

$$\frac{\sin\theta \sin(2k-1)\theta}{\cos^2 k\theta \cos^2(k-1)\theta} = \tan^2 k\theta - \tan^2(k-1)\theta. \tag{5}$$

Proof. Consider the following trigonometric identities

$$\sin k\theta \cos(k-1)\theta - \cos k\theta \sin(k-1)\theta = \sin\theta,$$
$$\sin k\theta \cos(k-1)\theta + \cos k\theta \sin(k-1)\theta = \sin(2k-1)\theta,$$

and divide the results by $\cos k\theta \cos(k-1)\theta$. This gives

$$\frac{\sin\theta}{\cos k\theta \cos(k-1)\theta} = \tan k\theta - \tan(k-1)\theta,$$
$$(6)$$
$$\frac{\sin(2k-1)\theta}{\cos k\theta \cos(k-1)\theta} = \tan k\theta + \tan(k-1)\theta.$$

Multiplying the above two relations (6), we get (5). □

Lemma 2. *For k a positive integer and θ a real variable such that $0 < k\theta < \pi/2$, the following holds*

$$\frac{\sin(2k-1)\theta[\sin^2\theta + \sin^2(2k-1)\theta]}{\cos^4 k\theta \cos^4(k-1)\theta} = 2\csc\theta(\tan^4 k\theta - \tan^4(k-1)\theta). \tag{7}$$

Proof. Square the first relation (6) and get

$$\frac{\sin^2\theta}{\cos^2 k\theta \cos^2(k-1)\theta} = \tan^2\theta + \tan^2(k-1)\theta - 2\tan k\theta \tan(k-1)\theta. \tag{8}$$

Next, we square the second relation (6) and get

$$\frac{\sin^2(2k-1)\theta}{\cos^2 k\theta \cos^2(k-1)\theta} = \tan^2\theta + \tan^2(k-1)\theta + 2\tan k\theta \tan(k-1)\theta. \tag{9}$$

Then add (8) and (9) to get

$$\frac{\sin^2\theta + \sin^2(2k-1)\theta}{\cos^2 k\theta \cos^2(k-1)\theta} = 2(\tan^2 k\theta + \tan^2(k-1)\theta). \tag{10}$$

Multiplication of (10) by (5) and division of the result by $\sin\theta$ then gives (7). □

Lemma 3. *Consider an arbitrary, well-defined sequence of functions or numbers,*

$$\{w_k : k = 1, 2, 3, \ldots\}$$

and, for any positive integer q, define the following sums

$$s_0(q) = \sum_{k=1}^{q} w_k; \qquad s_1(q) = \sum_{k=1}^{q} k w_k.$$

Then the following holds:

$$s_1(q) = (q+1)s_0(q) - \sum_{k=1}^{q} s_0(k). \tag{11}$$

Proof. Consider the following set of q equations:

$$\begin{aligned} w_1 + w_2 + w_3 + \cdots + w_q &= s_0(q); \\ w_2 + w_3 + \cdots + w_q &= s_0(q) - s_0(1); \\ w_3 + \cdots + w_q &= s_0(q) - s_0(2); \\ &\cdots \\ w_q &= s_0(q) - s_0(q-1). \end{aligned}$$

Then sum the terms in the left–hand side and equate the result to the sum of the terms in the right–hand side. This gives

$$w_1 + 2w_2 + \cdots + qw_q = qs_0(q) - \sum_{k=1}^{q-1} s_0(k).$$

Equation (11) follows upon adding and subtracting $s_0(q)$ in the right–hand side of this result. □

To get a closed form for a), divide (5) by $\sin\theta$ and note that the sum over $1 \le k \le q$ collapses. This gives:

$$P_0(q) = \sum_{k=1}^{q} \csc\theta(\tan^2 k\theta - \tan^2(k-1)\theta) = \csc\theta\tan^2 q\theta. \tag{12}$$

Similarly, use (7) to get a closed form for b) and get:

$$R_0(q) = \sum_{k=1}^{q} 2\csc\theta(\tan^4 k\theta - \tan^4(k-1)\theta) = 2\csc\theta\tan^4 q\theta. \tag{13}$$

We now find closed forms for $P_1(N)$ and $R_1(N)$, as defined in parts c) and d) of the problem statement.

To proceed, let q be an integer such that $1 \le k \le q$, with $0 < k\theta < \pi/2$, and consider the two functions

$$\begin{aligned} P_1(\theta; q) &= \sum_{k=1}^{q} \frac{k\sin(2k-1)\theta}{\cos^2 k\theta\cos^2(k-1)\theta} = (q+1)P_0(\theta, q) - \sum_{k=1}^{q} P_0(\theta, k); \\ R_1(\theta; q) &= \sum_{k=1}^{q} \frac{k\sin(2k-1)\theta(\sin^2\theta + \sin^2(2k-1)\theta)}{\cos^2 k\theta\cos^2(k-1)\theta} \\ &= (q+1)R_0(\theta, q) - \sum_{k=1}^{q} R_0(\theta, k). \end{aligned} \tag{14}$$

The right–hand sides of the two relations (14) follow from (11) applied to the pairs $\{P_1(\theta; q), P_0(\theta; q)\}$ and $\{R_1(\theta; q), R_0(\theta; q)\}$ with

$$\left\{ w_k = \frac{\sin(2k-1)\theta}{\cos^2 k\theta\cos^2(k-1)\theta} \right\} \quad \text{and} \quad \left\{ w_k = \frac{\sin(2k-1)\theta(\sin^2\theta + \sin^2(2k-1)\theta)}{\cos^4 k\theta\cos^4(k-1)\theta} \right\},$$

respectively. With $Q = \lfloor (N-1)/2 \rfloor$, as in the problem statement, sums c) and d) are given by

$$P_1(N) = P_1(\theta; q)\Big|_{\theta=\pi/N, q=Q} \qquad \text{and} \qquad R_1(N) = R_1(\theta, q)\Big|_{\theta=\pi/N, q=Q}. \tag{15}$$

Using (12) and (13) into (14) then gives, with the help of (15):

$$\begin{aligned} P_1(N) &= \csc\frac{\pi}{N}\left((Q+1)\tan^2\frac{Q\pi}{N} - \sum_{k=1}^{Q}\tan^2\frac{k\pi}{N}\right); \\ R_1(N) &= 2\csc\frac{\pi}{N}\left((Q+1)\tan^4\frac{Q\pi}{N} - \sum_{k=1}^{Q}\tan^4\frac{k\pi}{N}\right). \end{aligned} \tag{16}$$

We finally find $\sum_{k=1}^{Q}\tan^{2m}\frac{k\pi}{N}$ for $m = 1, 2$, by invoking the general results obtained in [1]. According to equation (31) of [1],

$$\sum_{k=1}^{Q}\tan^{2m}\frac{k\pi}{N} = \sum_{r=0}^{m}(-1)^{m-r}\binom{m}{r}S_{2r}(N), \qquad m = 1, 2, 3, \ldots$$

where

$$S_{2r}(N) = \sum_{k=1}^{Q}\sec^{2r}\frac{k\pi}{N}, \qquad r \geq 0.$$

For $r = 0$, we have

$$S_0(N) = Q,$$

and for $r = 1$ we have, from (26) and (27) of [1]:

$$S_r(N) = \begin{cases} \sum_{k=1}^{r} a_{k,r}(N^{2k} - 2^{2k}) & r \geq 1 \quad N \text{ even}, \\ \sum_{k=1}^{r}(2^{2k}-1)a_{k,r}(N^{2k}-1) & r \geq 1 \quad N \text{ odd}. \end{cases}$$

The $a_{k,r}$ coefficients that appear in these expressions are calculated as shown in [1]. For the cases of interest here, we need $a_{1,1} = \frac{1}{6}$, $a_{1,2} = \frac{1}{9}$, and $a_{2,2} = \frac{1}{90}$ (see second **Table**, p. 271 of [1]). These values give:

For N even,

$$S_0(N) = \frac{N-2}{2}, \qquad S_2(N) = a_{1,1}(N^2-4) = \frac{(N-2)(N+2)}{6},$$
$$S_4(N) = a_{1,2}(N^2-4) + a_{2,2}(N^4-16) = \frac{(N-2)(N+2)(N^2+14)}{90}.$$

For N odd,

$$S_0(N) = \frac{N-1}{2}, \qquad S_2(N) = 3a_{1,1}(N^2-1) = \frac{(N-1)(N+1)}{2},$$
$$S_4(N) = 3a_{1,2}(N^2-1) + 15a_{2,2}(N^4-1) = \frac{(N-1)(N+1)(N^2+3)}{6}.$$

Terms are arranged to highlight common factors. Next collect terms and factorize to get

$$\sum_{k=1}^{Q} \tan^2 \frac{k\pi}{N} = S_2(N) - S_0(N) = \begin{cases} \frac{(N-1)(N-2)}{6} & N \text{ even}, \\ \frac{N(N-1)}{2} & N \text{ odd}, \end{cases}$$

$$\sum_{k=1}^{Q} \tan^4 \frac{k\pi}{N} = S_4(N) - 2S_2(N) + S_0(N) = \begin{cases} \frac{(N-1)(N-2)(N^2+3N-13)}{90} & N \text{ even}, \\ \frac{N(N-1)(N^2+N-3)}{6} & N \text{ odd}. \end{cases}$$

These results can now be inserted in (16) to provide the sought closed forms and we find:

$$P_1(N) = \begin{cases} \csc \frac{\pi}{N} \left(\frac{N}{2} \tan^2 \frac{(N-2)\pi}{2N} - \frac{(N-1)(N-2)}{6} \right) & N \text{ even}, \\ \csc \frac{\pi}{N} \left(\frac{N+1}{2} \tan^2 \frac{(N-1)\pi}{2N} - \frac{N(N-1)}{2} \right) & N \text{ odd}, \end{cases}$$

$$R_1(N) = \begin{cases} 2\csc \frac{\pi}{N} \left(\frac{N}{2} \tan^4 \frac{(N-2)\pi}{2N} - \frac{(N-1)(N-2)(N^2+3N-13)}{90} \right) & N \text{ even}, \\ 2\csc \frac{\pi}{N} \left(\frac{N+1}{2} \tan^4 \frac{(N-1)\pi}{2N} - \frac{N(N-1)(N^2+N-3)}{6} \right) & N \text{ odd}. \end{cases}$$

This completes the proof of the problem.

References

[1] N. Gauthier and Paul S. Bruckman, *Sums of the even integral powers of the cosecant and secant*, The Fibonacci Quarterly, **44.3** (2006), 264–273.

Also solved by Paul S. Bruckman.

Errata: In the solution to H-690, the expression

$$(-1)^{k(m+1)} \sum_{k=1}^{n} \left\{ F_k^{2m} L_m + \sum_{i=1}^{\lfloor m/2 \rfloor} \sum_{r=1}^{i} \frac{m}{i} \binom{m-i-1}{i-1} \binom{i}{r} (-1)^{kr} F_k^{2(m-r)} \right\}$$

should be

$$\sum_{k=1}^{n} \left\{ (-1)^{k(m+1)} F_k^{2m} L_m + \sum_{i=1}^{\lfloor m/2 \rfloor} \sum_{r=1}^{i} \frac{m}{i} \binom{m-i-1}{i-1} \binom{i}{r} (-1)^{k(m+r+1)} F_k^{2(m-r)} \right\}.$$