ADVANCED PROBLEMS AND SOLUTIONS

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PROBLEMS PROPOSED IN THIS ISSUE

H-747 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Let $\binom{n}{k}_F$ denote the Fibonomial coefficient. For positive integer n, find closed form expressions for the sums:

(i) $\sum_{k=0}^{n-1}(-1)^k F_{2k}^2 \left(L_{k+1}L_{k+2}\cdots L_n\right)^2 \binom{2k}{k}_F$;

(ii) $\sum_{k=0}^{n-1}(-1)^k F_{4k+1} \left(L_{k+1}L_{k+2}\cdots L_n\right)^4 \binom{2k}{k}_F^2$.

H-748 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.

Let $x_k = L_k$, $y_k = F_k$, $k = 1, \ldots, m$, $x_{m+1} = x_1$, $y_{m+1} = y_1$. Prove that:

$$\frac{2}{F_{n+2}} + \sum_{k=1}^{m} \frac{x_k^3}{F_{n+1}x_k + F_n x_{k+1}} \geq \frac{L_m L_{m+1}}{F_{n+2}};$$

$$\sum_{k=1}^{m} \frac{y_k^3}{L_m y_k + L_{m+1} y_{k+1}} \geq \frac{F_m F_{m+1}}{L_{m+2}}$$

for every positive integer n.

H-749 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Let a, b, c and d be odd positive integers. If $a + b = c + d$, prove that

$$\sum_{k=1}^{b} \frac{F_a}{F_k F_{k+a}} + \sum_{k=1}^{a} \frac{F_b}{F_k F_{k+b}} = \sum_{k=1}^{d} \frac{F_c}{F_k F_{k+c}} + \sum_{k=1}^{c} \frac{F_d}{F_k F_{k+d}}.$$

H-750 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Generalized Tribonacci sequences $\{R_n\}$ and $\{S_n\}$ are defined by

$$R_{n+3} = pR_{n+2} + qR_{n+1} + rR_n \qquad \text{(for } n \geq 0\text{);}$$
$$S_{n+3} = pS_{n+2} + qS_{n+1} + rS_n \qquad \text{(for } n \geq 0\text{)}$$

with arbitrary $p, q, r, R_0, R_1, R_2, S_0, S_1, S_2$. For positive integers a, b, c, d such that $a+b = c+d$, prove that

$$R_{a+3}S_{b+3}+qR_{a+2}S_{b+2}+prR_{a+1}S_{b+1}-r^2R_aS_b = R_{c+3}S_{d+3}+qR_{c+2}S_{d+2}+prR_{c+1}S_{d+1}-r^2R_cS_d.$$

SOLUTIONS

Pell-Mell of Identities From Fibonacci-Lucas Polynomials

H-720 Proposed by N. Gauthier, The Royal Military College of Canada, Kingston, ON. (Vol. 50, No. 3, August 2012)

Let $\lfloor \ldots \rfloor$ be the largest integer function and, for a positive integer n, define $\varepsilon_n = 1$ for n even and $\varepsilon_n = 0$ for n odd. Then, with P_n the nth Pell number, prove the following identities:

(a) $\sum_{k\geq 0} \frac{\binom{n-2k}{2k}}{25^k} = \frac{1}{5^{n/2}6}\Big[\varepsilon_n(L_{2n+2} + 3L_{n+1}) + (1-\varepsilon_n)\sqrt{5}(F_{2n+2} + 3F_{n+1})\Big]$;

(b) $\sum_{k\geq 0} \frac{\binom{n-1-2k}{2k}}{16^k} = \frac{1}{2^n}[P_n + n]$;

(c) $\sum_{k=0}^{\lfloor (n-1)/4 \rfloor} \frac{\binom{n-1-2k}{2k}}{25^k(n-4k)} = \frac{1}{5^{n/2}n}\Big[\varepsilon_n(L_{2n} + L_n - 2(1 + (-1)^{n/2})) + (1-\varepsilon_n)\sqrt{5}(F_{2n} + F_n)\Big]$;

(d) $\sum_{k\geq 1} \frac{k\binom{n-1-k}{k}}{5^k} = \frac{1}{5^{n/2}54}\Big[\varepsilon_n((45n-20)F_{2n} - 15nL_{2n})$
$+ (1-\varepsilon_n)\sqrt{5}((9n-4)L_{2n} - 15nF_{2n})\Big]$.

Solution by Paul Bruckman.

Let

$$S_n(x) = \sum_{k=0}^{\lfloor n/4 \rfloor} \binom{n-2k}{2k} x^{2k} \quad \text{and} \quad G(y,x) = \sum_{n=0}^{\infty} S_n(x) y^n.$$

Then,

$$\begin{aligned}
G(y,x) &= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \binom{n+2k}{n} x^{2k} y^{n+4k} = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \binom{-2k-1}{n} (-y)^n (x^2y^4)^k \\
&= \sum_{k=0}^{\infty} (x^2y^4)^k (1-y)^{-2k-1} = (1-y)^{-1}\left(1 - \frac{x^2y^4}{(1-y)^2}\right)^{-1} \\
&= \frac{1-y}{(1-y)^2 - x^2y^4} = \frac{1}{2}\left(\frac{1}{1-y-xy^2} + \frac{1}{1-y+xy^2}\right) \\
&= \left(\frac{1+u}{4u}\right)\left(1 - y\left(\frac{1+u}{2}\right)\right)^{-1} - \left(\frac{1-u}{4u}\right)\left(1 - y\left(\frac{1-u}{2}\right)\right)^{-1} \\
&\quad + \left(\frac{1+v}{4v}\right)\left(1 - y\left(\frac{1+v}{2}\right)\right)^{-1} - \left(\frac{1-v}{4v}\right)\left(1 - y\left(\frac{1-v}{2}\right)\right)^{-1},
\end{aligned}$$

where $u = \sqrt{1+4x}$, $v = \sqrt{1-4x}$. Then,

$$S_n(x) = \frac{(1+u)^{n+1}}{2^{n+2}u} - \frac{(1-u)^{n+1}}{2^{n+2}u} + \frac{(1+v)^{n+1}}{2^{n+2}v} - \frac{(1-v)^{n+1}}{2^{n+2}v}. \tag{1}$$

Part (a). We seek $S_n(1/5)$. Setting $x = 1/5$ in (1), we obtain

$$u = \frac{3}{\sqrt{5}}, \quad v = \frac{1}{\sqrt{5}}, \quad 1+u = \frac{2\alpha^2}{\sqrt{5}}, \quad 1-u = \frac{-2\beta^2}{\sqrt{5}}, \quad 1+v = \frac{2\alpha}{\sqrt{5}}, \quad 1-v = \frac{-2\beta}{\sqrt{5}}.$$

Then, after simplification,

$$S_n(1/5) = \frac{\alpha^{2n+2} + (-1)^n\beta^{2n+2}}{6 \cdot 5^{n/2}} + \frac{\alpha^{n+1} + (-1)^n\beta^{n+1}}{2 \cdot 5^{n/2}}.$$

If n is even, then $S_n(1/5) = (L_{2n+2} + 3L_{n+1})/(6 \cdot 5^{n/2})$, while if n is odd, we then have $S_n(1/5) = (F_{2n+2} + 3F_{n+1})/(6 \cdot 5^{(n-1)/2})$. We may remove the dichotomy with the following formula valid for all n:

$$S_n(1/5) = \left(\frac{L_{2n+2} + 3L_{n+1}}{6 \cdot 5^{n/2}}\right)\varepsilon_n + \sqrt{5}\left(\frac{F_{2n+2} + 3F_{n+1}}{6 \cdot 5^{n/2}}\right)(1-\varepsilon_n).$$

Part (b). We seek $S_{n-1}(1/4)$. Setting $x = 1/4$ in (1) and substituting $n-1$ for n, we obtain

$$u = \sqrt{2}, \quad v = 0, \quad 1+u = 1+\sqrt{2}, \quad 1-u = 1-\sqrt{2}, \quad 1+v = 1-v = 1.$$

Note that the terms involving v in (1) for $v = 0$ reduce to the following:

$$\lim_{v \to 0}\left(\frac{1}{2^{n+1}v}\left(1 + nv - 1 + nv + \text{terms involving } v^2 \text{ and higher powers of } v\right)\right) = \frac{n}{2^n}.$$

Then

$$S_{n-1}(1/4) = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2^{n+1}\sqrt{2}} + \frac{n}{2^n} = \frac{P_n + n}{2^n}.$$

Part (c). Let

$$T_n(x) = \sum_{k=0}^{\lfloor (n-1)/4 \rfloor} \binom{n-1-2k}{2k} \frac{x^{n-4k}}{n-4k} \qquad (n \geq 1).$$

Then

$$T_n'(x) = \sum_{k=0}^{\lfloor (n-1)/4 \rfloor} \binom{n-1-2k}{2k} x^{n-4k-1} = x^{n-1}S_{n-1}(x^{-2}).$$

Using the expression (1), we obtain after simplification:

$$T_n'(x) = \frac{(x+\sqrt{x^2+4})^n - (x-\sqrt{x^2+4})^n}{2^{n+1}\sqrt{x^2+4}} + \frac{(x+\sqrt{x^2-4})^n - (x-\sqrt{x^2-4})^n}{2^{n+1}\sqrt{x^2-4}}.$$

Note that $T(0) = 0$. Then we find the following:

$$T_n(x) = \frac{(x+\sqrt{x^2+4})^n + (x-\sqrt{x^2+4})^n}{2^{n+1}n} + \frac{(x+\sqrt{x^2-4})^n + (x-\sqrt{x^2-4})^n}{2^{n+1}n} - \frac{2\varepsilon_n\varepsilon_{n/2}}{n}. \tag{2}$$

We seek $5^{-n/2}T_n(5^{1/2})$. Then

$$5^{-n/2}T_n(5^{1/2}) = \frac{(3+\sqrt{5})^n + (-1)^n(3-\sqrt{5})^n + (1+\sqrt{5})^n + (-1)^n(1-\sqrt{5})^n}{n2^{n+1}5^{n/2}} - \frac{2\varepsilon_n\varepsilon_{n/2}}{n5^{n/2}}$$
$$= \frac{\alpha^{2n} + (-1)^n\beta^{2n} + \alpha^n + (-1)^n\beta^n}{2n5^{n/2}} - \frac{2\varepsilon_n\varepsilon_{n/2}}{n5^{n/2}}.$$

If n is odd, then $5^{-n/2}T_n(5^{1/2}) = \sqrt{5}(F_{2n}+F_n)/(2n5^{n/2})$, while if n is even, then we have $5^{-n/2}T_n(5^{1/2}) = (L_{2n}+L_n-4\varepsilon_{n/2})/(2n5^{n/2})$. Therefore,

$$5^{-n/2}T_n(5^{1/2}) = \frac{1}{2n5^{n/2}}\left(\varepsilon_n(L_{2n}+L_n-4\varepsilon_{n/2}) + (1-\varepsilon_n)\sqrt{5}(F_{2n}+F_n)\right).$$

This differs from the answer given in the statement of the problem by a factor of 2 in the denominator.

Part (d). Let

$$U_n(x) = \sum_{k=1}^{\lfloor (n-1)/2\rfloor}\binom{n-1-k}{k}kx^k.$$

We seek $U_n(1/5)$. We note that

$$\int_0^x \frac{U_n(t)}{t}dt = \sum_{k=1}^{\lfloor (n-1)/2\rfloor}\binom{n-1-k}{k}x^k = \Phi_n(x),$$

where

$$\Phi_m(x) = \frac{p^m-q^m}{p-q}, \quad p = \frac{1+\sqrt{1+4x}}{2}, \quad q = \frac{1-\sqrt{1+4x}}{2}.$$

We then see that

$$\frac{U_n(x)}{x} = \Phi_n'(x) = \frac{n\Lambda_{n-1}(x) - 2\Phi_n(x)}{1+4x},$$

where $\Lambda_m(x) = p^m + q^m$. In particular,

$$U_n(1/5) = \frac{1}{5}\times\frac{5}{9}\left(n\Lambda_{n-1}(1/5) - 2\Phi_n(1/5)\right).$$

Note that

$$p(1/5) = \frac{\sqrt{5}+3}{2\sqrt{5}} = \frac{\alpha^2}{\sqrt{5}} \quad \text{and} \quad q(1/5) = \frac{\sqrt{5}-3}{2\sqrt{5}} = -\frac{\beta^2}{\sqrt{5}}.$$

Then

$$\Lambda_m(1/5) = \frac{\alpha^{2m} + (-1)^m\beta^{2m}}{5^{m/2}} \quad \text{and} \quad \Phi_m(1/5) = \frac{\sqrt{5}(\alpha^{2m} - (-1)^m\beta^{2m})}{3\cdot 5^{m/2}}.$$

Then

$$U_n(1/5) = \frac{1}{9}\left(n\left(\frac{\alpha^{2n-2} - (-1)^n\beta^{2n-2}}{5^{(n-1)/2}}\right) - 2\sqrt{5}\left(\frac{\alpha^{2n} - (-1)^n\beta^{2n}}{3\cdot 5^{n/2}}\right)\right). \tag{3}$$

If n is even,

$$U_n(1/5) = \frac{5}{27\cdot 5^{n/2}}(3nF_{2n-2} - 2F_{2n}).$$

Since $F_{2n-2} = (3F_{2n} - L_{2n})/2$, then (if n is even),

$$U_n(1/5) = \frac{5}{54 \cdot 5^{n/2}} \left(3n(3F_{2n} - L_{2n}) - 4F_{2n}\right) = \frac{5}{54 \cdot 5^{n/2}} \left((9n-4)F_{2n} - 3nL_{2n}\right)$$
$$= \frac{1}{54 \cdot 5^{n/2}} \left((45n-20)F_{2n} - 15nL_{2n}\right).$$

If n is odd, then from (3),

$$U_n(1/5) = \frac{\sqrt{5}}{27 \cdot 5^{n/2}} \left(3nL_{2n-2} - 2L_{2n}\right).$$

Since $L_{2n-2} = (3L_{2n} - 5F_{2n})/2$, then (if n is odd),

$$U_n(1/5) = \frac{\sqrt{5}}{54 \cdot 5^{n/2}} \left(3n(3L_{2n} - 5F_{2n}) - 4L_{2n}\right) = \frac{\sqrt{5}}{54 \cdot 5^{n/2}} \left((9n-4)L_{2n} - 15nF_{2n}\right).$$

Therefore, for all n,

$$U_n(1/5) = \frac{1}{54 \cdot 5^{n/2}} \left(\varepsilon_n(45n-20)F_{2n} - 15nL_{2n}) + (1-\varepsilon_n)\sqrt{5}((9n-4)L_{2n} - 15nF_{2n}\right).$$

Also solved by Kenny B. Davenport and the proposer.

A Problem for Fibonacci and Harmonic Numbers

H-721 Proposed by Khristo N. Boyadzhiev, Ohio Northern University, Ada, Ohio.
(Vol. 50, No. 3, August 2012)

Let $H_0 = 0$ and $H_n = 1 + 1/2 + \cdots + 1/n$ for $n \geq 1$ be the harmonic numbers. Show that

$$\sum_{n=0}^{\infty} F_n H_n z^n = C(z) \sum_{n=0}^{\infty} F_n z^n, \qquad \text{where} \qquad C(z) = 1 + \sum_{n=1}^{\infty} \left(\frac{F_{n-1}}{n} + \frac{F_{n+1}}{n+1}\right) z^n.$$

Solution by Robinson Higuita, Universidad de Antioquia, Medellin, Colombia.

Note that

$$H_{n+2}F_{n+2} - H_{n+1}F_{n+1} - H_nF_n = H_nF_{n+2} + \frac{F_{n+2}}{n+1} + \frac{F_{n+2}}{n+2} - H_nF_{n+1} - \frac{F_{n+1}}{n+1} - H_nF_n$$
$$= (F_{n+2} - F_{n+1} - F_n)H_n + \frac{F_{n+2} - F_{n+1}}{n+1} + \frac{F_{n+2}}{n+2}$$
$$= \frac{F_n}{n+1} + \frac{F_{n+2}}{n+2}. \qquad (4)$$

Let $g(z) = \sum_{k=0}^{\infty} H_k F_k z^k$ be the generating function of the $\{H_n F_n\}$ sequence, then

$$\begin{aligned} g(z)(1-z-z^2) &= \sum_{k=0}^{\infty} H_k F_k z^k - \sum_{k=0}^{\infty} H_k F_k z^{k+1} - \sum_{k=0}^{\infty} H_k F_k z^{k+2} \\ &= \sum_{k=3}^{\infty} H_k F_k z^k - \sum_{k=2}^{\infty} H_k F_k z^{k+1} - \sum_{k=1}^{\infty} H_k F_k z^{k+2} + z + H_2 z^2 - H_1 z^2 \\ &= \sum_{k=1}^{\infty} H_{k+2} F_{k+2} z^{k+2} - \sum_{k=1}^{\infty} H_{k+1} F_{k+1} z^{k+2} - \sum_{k=1}^{\infty} H_k F_k z^{k+2} + z + \frac{z^2}{2} \\ &= \sum_{k=1}^{\infty} \left(H_{k+2}F_{k+2} - H_{k+1}F_{k+1} - H_k F_k\right) z^{k+2} + z + \frac{z^2}{2}. \end{aligned}$$

This and (4) imply that

$$\begin{aligned} g(z)(1-z-z^2) &= \sum_{k=1}^{\infty} \left(\frac{F_k}{k+1} + \frac{F_{k+2}}{k+2}\right) z^{k+2} + z + \frac{z^2}{2} \\ &= z\left(1 + \sum_{k=1}^{\infty} \left(\frac{F_{k-1}}{k} + \frac{F_{k+1}}{k+1}\right) z^k\right) = zC(z). \end{aligned}$$

As

$$\sum_{k=1}^{\infty} F_n z^n = \frac{z}{1-z-z^2}$$

(see [1], page 220), then

$$\sum_{k=0}^{\infty} F_k H_k z^k = g(z) = \frac{z}{1-z-z^2} C(z) = C(z) \sum_{k=1}^{\infty} F_n z^n.$$

References

[1] T. Koshy, *Fibonacci and Lucas numbers with Applications*, John Wiley, New York, 2001.

Also solved by Paul S. Bruckman, Kenneth B. Davenport, Dmitry Fleischman, Anastasios Kotronis, and the proposer.

A Trigonometric Series

H-722 Proposed by Ovidiu Furdui, Campia Turzii, Romania. (Vol. 50, No. 3, August 2012)

Let $x \in (0, 2\pi)$, $k \geq 1$ be a natural number and

$$S_k(x) = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n(n+1)(n+2)\cdots(n+k)}.$$

Prove that $S_k(x)$ equals

$$\frac{(2\sin(\frac{x}{2}))^k}{k!}\left(-\cos\frac{(\pi-x)k}{2}\cdot\frac{\ln(2(1-\cos x))}{2}-\frac{\pi-x}{2}\sin\frac{(\pi-x)k}{2}+\sum_{j=1}^{k}\frac{\cos\frac{(\pi-x)(j-k)}{2}}{j(2\sin(x/2))^j}\right).$$

Solution by Robinson Higuita, Universidad de Antioquia, Medellin, Colombia.

Let $z=\cos y+i\sin y$ then $z^n=\cos ny+i\sin ny$. Note that

$$S_k(y)=\mathrm{Re}\left(\sum_{n=1}^{\infty}\frac{z^n}{n(n+1)\dots(n+k)}\right).$$

As

$$\frac{1}{n(n+1)\dots(n+k)}=\frac{1}{k!}\frac{\Gamma(k+1)\Gamma(n)}{\Gamma(n+k+1)}=\frac{1}{k!}\int_0^1(1-x)^k x^{n-1}dx$$

(see [1], Page 41), we have that

$$\begin{aligned}S_k(y)&=\frac{1}{k!}\mathrm{Re}\left(\sum_{n=1}^{\infty}z\int_0^1(1-x)^k(zx)^{n-1}dx\right)=\frac{1}{k!}\mathrm{Re}\left(z\int_0^1(1-x)^k\sum_{n=1}^{\infty}(zx)^{n-1}dx\right)\\&=\frac{1}{k!}\mathrm{Re}\left(\int_0^1(1-x)^k\frac{z}{1-zx}dx\right)=\frac{1}{k!}\mathrm{Re}\left(\int_0^1\frac{(1-x)^k\bar{z}z}{\bar{z}-z\bar{z}x}dx\right)\\&=\frac{1}{k!}\mathrm{Re}\left(\int_0^1\frac{(1-x)^k}{\bar{z}-x}dx\right).\end{aligned}$$

Using the following identity

$$\int_0^1\frac{(1-x)^{k+1}}{\bar{z}-x}dx=\int_0^1(1-x)^k\left(\frac{1-\bar{z}}{\bar{z}-x}+1\right)dx$$

and mathematical induction, we can prove that

$$\int_0^1\frac{(1-x)^k}{\bar{z}-x}dx=\sum_{j=1}^{k}\frac{(1-\bar{z})^{k-j}}{j}-(1-\bar{z})^k(\ln(\bar{z}-1)-\ln\bar{z}).$$

Therefore,

$$S_k(y)=\frac{1}{k!}\mathrm{Re}\left(\int_0^1\frac{(1-x)^k}{\bar{z}-x}dx\right)=\frac{1}{k!}\mathrm{Re}\left(\sum_{j=1}^{k}\frac{(1-\bar{z})^{k-j}}{j}-(1-\bar{z})^k(\ln(\bar{z}-1)-\ln\bar{z})\right). \quad (5)$$

As $y\in(0,2\pi)$, then

$$\ln(\bar{z}-1)-\ln\bar{z}=\ln|e^{-iy}-1|+i\arg(e^{-iy}-1)-i\arg(e^{-iy})=\frac{\ln 2(1-\cos y)}{2}-\frac{(\pi-y)}{2}i. \quad (6)$$

On the other hand,

$$\begin{aligned}\operatorname{Re}((1-e^{-yi})^m) &= \frac{(1-e^{-yi})^m+(1-e^{yi})^m}{2}\\ &= \frac{1}{2}\left(2e^{\frac{\pi}{2}i}e^{\frac{-y}{2}i}\frac{e^{\frac{yi}{2}}-e^{\frac{-yi}{2}}}{2i}\right)^m+\frac{1}{2}\left(2e^{\frac{-\pi}{2}i}e^{\frac{y}{2}i}\frac{e^{\frac{yi}{2}}-e^{\frac{-yi}{2}}}{2i}\right)^m\\ &= \left(2\frac{e^{\frac{yi}{2}}-e^{\frac{-yi}{2}}}{2i}\right)^m\left(\frac{e^{\frac{\pi-y}{2}mi}+e^{-\frac{\pi-y}{2}mi}}{2}\right)\\ &= \left(2\sin\frac{y}{2}\right)^m\cos\frac{\pi-y}{2}m \qquad (7)\end{aligned}$$

for all $m\in\mathbb{Z}$.

Similarly, we can prove that

$$\operatorname{Re}(i(1-e^{-yi})^m) = -\left(2\sin\frac{y}{2}\right)^m\sin\left(\frac{\pi-y}{2}m\right)\quad\text{for all}\quad m\in\mathbb{Z}. \qquad (8)$$

Replacing (6), (7) and (8) in (5), we have that $S_k(y)$ equals

$$\begin{aligned}&\frac{1}{k!}\left(\sum_{j=1}^{k}\frac{\operatorname{Re}((1-\bar z)^{k-j})}{j}-\frac{\ln 2(1-\cos y)}{2}\operatorname{Re}((1-\bar z)^k)+\frac{(\pi-y)}{2}\operatorname{Re}(i(1-\bar z)^k)\right)\\ &=\frac{1}{k!}\left(\sum_{j=1}^{k}\frac{\left(2\sin\frac{y}{2}\right)^k\cos\frac{(\pi-y)(k-j)}{2}}{j\left(2\sin\frac{y}{2}\right)^j}-\left(2\sin\frac{y}{2}\right)^k\cos\frac{\pi-y}{2}k.\frac{\ln 2(1-\cos y)}{2}\right)\\ &\quad-\frac{1}{k!}\left(\frac{(\pi-y)}{2}\left(2\sin\frac{y}{2}\right)^k\sin\frac{\pi-y}{2}k\right)\\ &=\frac{(2\sin\frac{y}{2})^k}{k!}\left(-\cos\frac{(\pi-y)k}{2}.\frac{\ln 2(1-\cos y)}{2}-\frac{\pi-y}{2}\sin\frac{(\pi-y)k}{2}+\sum_{j=1}^{k}\frac{\cos\frac{(\pi-y)(j-k)}{2}}{j(2\sin(y/2))^j}\right).\end{aligned}$$

References

[1] W. J. Kaczor, *Problems in Mathematical Analysis III*, American Mathematical Society, 2000.

Also solved by Paul S. Bruckman, Kenneth B. Davenport, and the proposer.

The Limit of a Multiple Sum

H-723 Proposed by Ovidiu Furdui, Campia Turzii, Romania. (Vol. 50, No. 3, August 2012)

Let $k\geq 2$ be an integer and let m be a nonnegative integer. Prove that

$$\lim_{n\to\infty}\frac{1}{n^{k-1}}\sum_{i_1=1}^{n}\cdots\sum_{i_k=1}^{n}\frac{1}{i_1+i_2+\cdots+i_k+m}=\frac{k}{(k-1)!}\sum_{j=2}^{k}(-1)^{k-j}j^{k-2}\binom{k-1}{j-1}\ln j.$$

Solution by the proposer.

We note that the limit equals

$$\int_0^1 \cdots \int_0^1 \frac{dx_1 dx_2 \cdots dx_k}{x_1 + x_2 + \cdots + x_k}.$$

Now we calculate the multiple integral. We have, since $1/a = \int_0^\infty e^{-at} dt$, that

$$\begin{aligned}
\int_0^1 \cdots \int_0^1 \frac{dx_1 dx_2 \cdots dx_k}{x_1 + x_2 + \cdots + x_k} &= \int_0^1 \cdots \int_0^1 \left(\int_0^\infty e^{-(x_1+x_2+\cdots+x_k)t} dt \right) dx_1 dx_2 \cdots dx_k \\
&= \int_0^\infty \left(\left(\int_0^1 e^{-x_1 t} dx_1 \right) \left(\int_0^1 e^{-x_2 t} dx_2 \right) \cdots \left(\int_0^1 e^{-x_k t} dx_k \right) \right) dt \\
&= \int_0^\infty \left(\frac{1-e^{-t}}{t} \right)^k dt = \int_0^\infty \left(\frac{1-e^{-x}}{x} \right)^k dx.
\end{aligned}$$

Now, since $1/x^k = \frac{1}{(k-1)!} \int_0^\infty e^{-xt} t^{k-1} dt$, we obtain that

$$\begin{aligned}
\int_0^\infty \frac{(1-e^{-x})^k}{x^k} dx &= \frac{1}{(k-1)!} \int_0^\infty (1-e^{-x})^k \left(\int_0^\infty e^{-xt} t^{k-1} dt \right) dx \\
&= \frac{1}{(k-1)!} \int_0^\infty t^{k-1} \left(\int_0^\infty (1-e^{-x})^k e^{-xt} dx \right) dt \\
&= \frac{(-1)^k}{(k-1)!} \int_0^\infty t^{k-1} \left(\int_0^\infty (e^{-x}-1)^k e^{-xt} dx \right) dt \\
&= \frac{(-1)^k}{(k-1)!} \int_0^\infty t^{k-1} \left(\int_0^\infty \sum_{j=0}^k \binom{k}{j} e^{-x(j+t)} (-1)^{k-j} dx \right) dt.
\end{aligned}$$

Since, $(-1)^{k-j} = (-1)^{k+j}$, we get that

$$\int_0^\infty \frac{(1-e^{-x})^k}{x^k} dx = \frac{1}{(k-1)!} \int_0^\infty t^{k-1} \left(\sum_{j=0}^k (-1)^j \binom{k}{j} \frac{1}{t+j} \right) dt.$$

A calculation shows that

$$\frac{1}{t(t+1)\cdots(t+k)} = \sum_{j=0}^k \frac{a_j}{t+j}, \quad \text{where} \quad a_j = \frac{(-1)^j \binom{k}{j}}{k!},$$

and hence,

$$\int_0^\infty \frac{(1-e^{-x})^k}{x^k} dx = k \int_0^\infty \frac{t^{k-2}}{(t+1)(t+2)\cdots(t+k)} dt.$$

On the other hand,

$$\frac{t^{k-2}}{(t+1)\cdots(t+k)} = \sum_{j=1}^k \frac{b_j}{t+j}, \quad \text{where} \quad b_j = \frac{(-1)^{k-j-1} j^{k-2}}{(k-1)!} \binom{k-1}{j-1},$$

and note that $b_1 + b_2 + \cdots + b_k = 0$. Thus,

$$\int_0^\infty \frac{t^{k-2}}{(t+1)(t+2)\cdots(t+k)}dt = \int_0^\infty \left(\sum_{j=1}^{k} \frac{b_j}{t+j}\right) dt = \left(\sum_{j=1}^{k} b_j \ln(t+j)\right)\Bigg|_{t=0}^{t=\infty}$$

$$= -\sum_{j=1}^{k} b_j \ln j,$$

since $\lim_{t\to\infty} \sum_{j=1}^{k} b_j \ln(t+j) = 0$. To prove this, we have, since $b_1 = -b_2 - \cdots - b_k$, that $\lim_{t\to\infty} \sum_{j=1}^{k} b_j \ln(t+j) = \lim_{t\to\infty} \sum_{j=2}^{k} b_j \ln \frac{t+j}{t+1} = 0$. It follows that the desired limit equals

$$\frac{k}{(k-1)!}\left(\sum_{j=2}^{k}(-1)^{k-j} j^{k-2}\binom{k-1}{j-1}\ln j\right),$$

and the problem is solved.

Also solved by Paul S. Bruckman.

Late acknowledgement. Problem **H-719** was also solved by the proposer.

Errata: The constant term in the right–hand side of **H-731** should be $-1/4\pi$ instead of $(3\pi - 12)/12\pi$.

Withdrawals. The proposer withdrew Problem **H-745** in March 2013, which shortly afterwards appeared as Problem 1278 in the *Pi Mu Epsilon* Journal. The editor apologizes for the oversight.