The sequences of the Fibonacci and Lucas polynomials are defined by

\[ F_0(x) = 0, \quad F_1(x) = 1, \quad \text{and} \quad F_{n+1}(x) = xF_n(x) + F_{n-1}(x) \text{ for } n \geq 1, \]

\[ L_0(x) = 2, \quad L_1(x) = x, \quad \text{and} \quad L_{n+1}(x) = xL_n(x) + L_{n-1}(x) \text{ for } n \geq 1, \]

respectively. Prove that, for all non-zero complex numbers \( x \) and all positive integers \( n \),

(a) \[
\sum_{k=0}^{2n-1} \binom{4n-1-k}{k} 2^{4n-1-2k} x^k F_k(x) = x^{2n-1} L_{2n-1}(x) F_{2n}(4/x),
\]

(b) \[
\sum_{k=0}^{2n-1} \binom{4n-1-k}{k} 2^{4n-1-2k} x^k L_k(x) = x^{2n-1} (x^2 + 4) F_{2n-1}(x) F_{2n}(4/x),
\]

(c) \[
\sum_{k=0}^{2n} \binom{4n+1-k}{k} 2^{4n+2-2k} x^k F_k(x) = x^{2n+1} F_{2n}(x) L_{2n+1}(4/x),
\]

(d) \[
\sum_{k=0}^{2n} \binom{4n+1-k}{k} 2^{4n+2-2k} x^k L_k(x) = x^{2n+1} L_{2n}(x) L_{2n+1}(4/x).
\]
H-640 Proposed by Jayantibhai M. Patel, Ahmedabad, India

For any positive integer \( n \geq 2 \), prove that the value of the following determinant

\[
\begin{vmatrix}
-L_{2n} & F_{2n} & L_n^2 & 2F_{2n} & L_n^2 \\
F_{2n} & -3(3F_n^2 + 2(-1)^n) & F_{2n}^2 & F_{2n}^2 & F_{2n} \\
L_n^2 & F_{2n} & -L_{2n} & 2F_{2n} & L_n^2 \\
2F_{2n} & 2F_n & 2F_{2n}^2 & -6F_{n+1}F_{n-1} & 2F_{2n} \\
L_n^2 & F_{2n} & L_n^2 & 2F_{2n} & -L_{2n}
\end{vmatrix}
\]

is \( (L_n^2 + L_{2n})^5 \).

H-641 Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY

A composition of \( n \) is an ordered sequence of positive integers having sum equal to \( n \). The terms of the sequence are called the parts of \( n \). It is known that the number of compositions of \( n \) is \( 2^n - 1 \) and the number of compositions of \( n \) with exactly \( k \) parts is equal to \( \binom{n-1}{k-1} \). Here, we consider a slightly modified concept assuming that there are two kinds of 1. Find:

(i) The number \( a_n \) of compositions of \( n \) (for example, \( a_2 = 5 \) because we have (2), (1,1), (1,1'), (1',1), and (1',1'));

(ii) the number \( c_{n,k} \) of compositions of \( n \) with exactly \( k \) parts (for example, \( c_{4,2} = 5 \) because we have (1,3), (1',3), (3,1), (3,1'), and (2,2)).

H-642 Walther Janous, Innsbruck, Austria

Determine the limit

\[
\lim_{n \to \infty} \left( \frac{L_n^2}{F_{n+2}} - \sum_{k=1}^{n} \frac{L_k^2}{F_k} \right).
\]

SOLUTIONS

Bounding ratios of Fibonacci numbers

H-625 Proposed by Russel Jay Hendel, Townson University, MD

(Vol. 3, no. 2, May 2005)

For an integer \( m > 0 \) let \( K_m \) be the smallest positive integer such that \( F_{n+m} < K_m F_n \) holds for all large \( n \). For example, \( K_1 = 2 \) because \( F_n < F_{n+1} < 2F_n \) holds for all large \( n \). Provide an explicit formula for \( K_m \).

Solution by H.-J. Seiffert

It is known (see equations (3.22) and (3.24) in [1]) that, for all integers \( m \) and \( n \),

\[
F_{n+m} + (-1)^m F_{n-m} = L_m F_n.
\]
We shall prove that, for $m > 0$,

$$K_m = \begin{cases} L_m + 1, & \text{if } m \text{ is odd}, \\ L_m, & \text{if } m \text{ is even}. \end{cases}$$

Suppose that $m$ is odd. If $n > m$, then, by (1),

$$L_m F_n = F_{n+m} - F_{n-m} < F_{n+m} < F_{n+m} + F_n - F_{n-m} = (L_m + 1)F_n.$$

Let $m$ be even. If $n > m$, then, by (1) again,

$$(L_m - 1)F_n = F_{n+m} - (F_n - F_{n-m}) < F_{n+m} < F_{n+m} + F_{n-m} = L_m F_n.$$

This completes the proof of the above statement.


Also solved by Paul S. Bruckman and the proposer.

**Pell numbers and Fibonacci polynomials**

H-626 Proposed by H.-J. Seiffert, Berlin, Germany
(Vol. 43, no. 2, May 2005)

The Fibonacci polynomials are defined by $F_0(x) = 0$, $F_1(x) = 1$, and $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$ for $n \geq 0$. Let $n$ be a positive integer.

a. Prove that, for all complex numbers $x$,

$$F_{n+1}(x) + iF_n(x) = 4^{-n} \sum_{k=0}^{n} \binom{2n + 1}{2k + 1} (x - 2i)^k (x + 2i)^{n-k}, \quad \text{where } i = \sqrt{-1}.$$

b. Deduce the identities

$$P_n = 2^{-\lfloor \frac{n}{2} \rfloor} \sum_{k=0}^{\lfloor \frac{n-2k}{4} \rfloor} (-1)^{\lfloor \frac{n-2k}{4} \rfloor} \binom{2n - 1}{2k + 1} \quad \text{and} \quad P_n = 2^{-\lfloor \frac{n+1}{2} \rfloor} \sum_{k=0}^{\lfloor \frac{n-2k+1}{4} \rfloor} (-1)^{\lfloor \frac{n-2k+1}{4} \rfloor} \binom{2n + 1}{2k + 1},$$

where $P_n = F_n(2)$ is the $n$th Pell number.

Solution by the proposer

189
It is well-known that
\[ F_{2n+1}(x) = \frac{1}{\sqrt{x^2 + 4}} \left( \left( \frac{x + \sqrt{x^2 + 4}}{2} \right)^{2n+1} - \left( \frac{x - \sqrt{x^2 + 4}}{2} \right)^{2n+1} \right). \]

Applying the Binomial Theorem gives
\[ F_{2n+1}(x) = 4^{-n} \sum_{k=0}^{n} \binom{2n+1}{2k+1} (x^2 + 4)^k x^{2n-2k}. \]

Replacing \( x \) by \( \sqrt{ix - 2} \) (here, \( \sqrt{ix - 2} \) can be any of the at most two possible square roots of \( ix - 2 \)) and multiplying by \( (-i)^n \) yields
\[ (-i)^n F_{2n+1}(\sqrt{ix - 2}) = 4^{-n} \sum_{k=0}^{n} \binom{2n+1}{2k+1} (x - 2i)^k (x + 2i)^{n-k}. \]

Using the known relation \( F_{2j}(y)/y = i^{1-j} F_j(i(y^2 + 2)) \) with \( y = \sqrt{ix - 2} \) and noting that \( F_j(-x) = (-1)^{j-1} F_j(x) \), we find
\[ (-i)^n F_{2n+1}(\sqrt{ix - 2}) = \frac{(-i)^n}{\sqrt{ix - 2}} \left( F_{2n+2}(\sqrt{ix - 2}) - F_{2n}(\sqrt{ix - 2}) \right) = F_{n+1}(x) + iF_n(x). \]

This proves the identity of part a.

Since \( 1 - i = \sqrt{2}e^{-i\pi/4} \) and \( 1 + i = \sqrt{2}e^{i\pi/4} \), the identity of a with \( x = 2 \) implies that
\[ P_{n+1} + iP_n = 2^{-n/2} \sum_{k=0}^{n} \binom{2n+1}{2k+1} \exp \left( \frac{(n - 2k)\pi i}{4} \right). \]

Using Euler’s relation \( e^{iy} = \cos y + i \sin y \), after equating the real and imaginary parts, we find
\[ P_{n+1} = 2^{-n/2} \sum_{k=0}^{n} \binom{2n+1}{2k+1} A_{n-2k}, \quad (1) \]

and
\[ P_n = 2^{-n/2} \sum_{k=0}^{n} \binom{2n+1}{2k+1} B_{n-2k}, \quad (2) \]
where $A_j = \cos(j\pi/4)$ and $B_j = \sin(j\pi/4)$, for $j \in \mathbb{Z}$. Simple calculations show that

$$A_j = \begin{cases} (-1)^{(j+1)/4}2^{[j/2]-j/2}, & \text{if } j \not\equiv 2 \pmod{4}, \\ 0, & \text{if } j \equiv 2 \pmod{4}, \end{cases}$$

$$B_j = \begin{cases} (-1)^{(j-1)/4}2^{[j/2]-j/2}, & \text{if } j \not\equiv 0 \pmod{4}, \\ 0, & \text{if } j \equiv 0 \pmod{4}. \end{cases}$$

Now the identities of part b follows from (1) with $n$ replaced by $n - 1$ and (2).

Also solved by Paul S. Bruckman.

**Revisiting the Cauchy-Schwartz inequality**

**H-627** Proposed by Slavko Simic, Belgrade, Yugoslavia  
(Vol. 43, no. 3, August 2005)

Find all sequences $c = \{c_i\}_{i=1}^n$, $c_i = c_i(n)$ such that the inequality

$$|x^* - \sum_{i=1}^n c_i x_i| \leq \sqrt{n - 1} \left( \sum_{i=1}^n c_i x_i^2 - \left( \sum_{i=1}^n c_i x_i \right)^2 \right),$$

holds for all sequences $x = \{x_i\}_{i=1}^n$ of arbitrary real numbers and arbitrary $x^* \in x$.

**Solution by the proposer**

We show that the conditions of the problem are satisfied if and only if $c_i = 1/n$ for $i = 1, \ldots, n$.

Putting $x_i = 1$ for all $i = 1, \ldots, n$, we see that a necessary condition for the given inequality to hold is $(\sum_{i=1}^n c_i)(1 - \sum_{i=1}^n c_i) \geq 0; \text{ i.e.},$

$$0 \leq \sum_{i=1}^n c_i \leq 1. \quad (1)$$

Also, putting subsequently $x_i = 0, i \neq s$, $x^*_s = x_s = 1$, for all $s = 1, \ldots, n$, in the desired inequality, we obtain

$$|1 - c_s| \leq \sqrt{n - 1}\sqrt{c_s(1 - c_s)}, \quad s = 1, \ldots, n;$$
ADVANCED PROBLEMS AND SOLUTIONS

\[ c_s \geq \frac{1}{n} \text{ for all } s = 1, \ldots, n. \] But these last inequalities together with (1) imply that \( c_s = \frac{1}{n} \) for all \( s = 1, \ldots, n \). This last condition is also sufficient since

\[
(n - 1) \left( \frac{\sum_{i=1}^{n} x_i^2}{n} - \left( \frac{\sum_{i=1}^{n} x_i}{n} \right)^2 \right) = \frac{n - 1}{n^2} \left( n \sum_{i=1}^{n} x_i^2 - \left( \sum_{i=1}^{n} x_i \right)^2 \right)
\]

\[
= \frac{n - 1}{n^2} \sum_{1 \leq i < j \leq n} (x_i - x_j)^2 \geq \frac{1}{n^2} \left( (n - 1) \sum_{i=1}^{n} (x_i - x^*)^2 \right) \geq \frac{1}{n^2} \left( \sum_{i=1}^{n} (x_i - x^*)^2 \right)
\]

\[
= \left( \frac{\sum_{i=1}^{n} x_i}{n} - x^* \right)^2,
\]

which completes the proof.

**Sums of three cubes**

**H-628 Proposed by Juan Pla, Paris, France**

**(Vol. 43, no. 3, August 2005)**

Let us consider the set \( S \) of all the sequences \( \{U_n\}_{n \geq 0} \) satisfying a second order linear recurrence

\[
U_{n+2} - aU_{n+1} + bU_n = 0,
\]

with both \( a \) and \( b \) rational integers, and having only integral values. Prove that for infinitely many of these sequences their general term \( U_n \) is a sum of three cubes of integers for any value of the subscript \( n \).

**Solution by the proposer**

The starting point is the following easy to prove identity

\[
(x + y + z)^3 - (x^3 + y^3 + z^3) = 3(x + y)(y + z)(z + x).
\]

Setting in (1): \( x = U_{n+2}, \ y = -aU_{n+1}, \ z = bU_n \), we obtain easily

\[
U_{n+2}^3 + (-aU_{n+1})^3 + (bU_n)^3 = -3abU_{n+2}U_{n+1}U_n.
\]

But since we have the classical relation

\[
U_{n+2}U_n - U_{n+1}^2 = b^n(U_2U_0 - U_1^2),
\]

after substitution of \( U_nU_{n+2} \) in the right hand side of (2) and simplifications we obtain

\[
U_{n+2}^3 - (-3ab + a^3)U_{n+1}^3 + (bU_n)^3 = -3ab^{n+1}(U_2U_0 - U_1^2)U_{n+1}.
\]

To have a sum of three cubes on the left hand side, we need only that \(-3ab + a^3\) be a cube, which is the case if we set \( a \) to be an arbitrary integer and then look for \( b \) such that \(-3b + a^2 = a^2c^3\), or, equivalently, \( b = a^2(1-c^3)/3 \), with an arbitrary integer \( c \). In order for \( b \) to be an integer, it suffices to impose that either \( a \) is a multiple of 3 or \( c \equiv 1 (\text{mod } 3) \). It is now easy to prove that the sequence whose general term appears in the right hand side of relation (3) does belong to the set \( S \).