ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
FLORIAN LUCA

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, IMATE, UNAM, AP. POSTAL 61-3 (XANGARI), CP 58 089, MORELIA, MICHOACAN, MEXICO, or by e-mail at fluca@matmor.unam.mx as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-669 Proposed by G. C. Greubel, Newport News, VA

Show that

$$\sum_{n=0}^{\infty}\left[\frac{1}{5n+1}+\frac{2}{5n+2}+\frac{\beta^2}{5n+3}+\frac{\beta}{5n+4}-\frac{\beta^2}{5n+5}\right]=\frac{\alpha^3}{2}\ln(2\beta^2)+\mu\alpha^2\tan^{-1}(\beta^2\sigma),$$

where $\mu=(-\sqrt{5}\beta)^{1/2}$ and $\sigma=(5\alpha^2)^{1/4}$.

H-670 Proposed by P. Bruckman, Sointula, Canada

Let $\begin{bmatrix} n \\ k \end{bmatrix}$ denote the standard Fibonomial coefficient $F_nF_{n-1}\cdots F_{n-k+1}/(F_1F_2\cdots F_k)$.

(a) Define the following sums:

$$A_n=\sum_{k=0}^{n}(-1)^{k(k+1)/2}\begin{bmatrix} n \\ k \end{bmatrix}F_k, \qquad B_n=\sum_{k=0}^{n}(-1)^{k(k-1)/2}\begin{bmatrix} n \\ k \end{bmatrix}.$$

Prove that for $n\geq 1$, $A_n=-F_nB_{n-1}$.

(b) Define

$$C_n=\sum_{k=0}^{n}(-1)^{k(k+1)/2}\begin{bmatrix} n \\ k \end{bmatrix}L_k.$$

Prove that $C_n=-(L_n-2)B_{n-1}$.

H-671 Proposed by G. C. Greubel, Newport News, VA

Let $\phi_n(x,y)$ be the bi-variate Fibonacci and Lucas polynomials. Find expansions for

$$\sum_{n=0}^{\infty}\binom{n+m}{n}\phi_{n+p}(x,y),$$

in the form

$$P(x,y)+\sum_{r=0}^{\lambda}\binom{\lambda}{r}Q_r(x,y),$$

where $P(x,y)$ and $Q_r(x,y)$ are general polynomials and λ is given by: (A) $m+1$, (B) p.

H-672 Proposed by J. L. Díaz-Barrero, Barcelona, Spain

Let n be a positive integer. Prove that

$$\sum_{k=1}^{n}\left(\frac{F_k}{1+L_k}\right)^2 \geq \frac{1}{F_nF_{n+1}}\left(\sum_{k=1}^{n}\frac{F_k^2}{1+L_k}\right)^2 \geq F_n^3F_{n+1}^3\left(\sum_{k=1}^{n}F_k^2(1+L_k)\right)^{-2}.$$

SOLUTIONS

Binomial Coefficients, Powers of 2 and Fibonacci Numbers

H-651 Proposed by H.-J. Seiffert, Berlin, Germany
(Vol. 45, No. 1, February 2007)

Prove that, for all positive integers n,

$$\sum_{k=0}^{\lfloor (n-3)/5\rfloor}\binom{4n}{2n-10k-5} = \frac{1}{10}(2^{4n-1} - 5^nL_{2n} + L_{4n})$$

and

$$\sum_{k=0}^{\lfloor (n-3)/5\rfloor}\binom{4n-2}{2n-10k-6} = \frac{1}{10}(2^{4n-3} - 5^nF_{2n-1} + L_{4n-2}).$$

Solution by Paul S. Bruckman, Sointula, Canada

Suppose that $n \geq 3$. Let A_n and B_n denote the first and second sums appearing in the statement of the problem. Observe that both expressions are special cases of the quantity

$$H_m = \sum_{k=0}^{\lfloor (m-5)/10\rfloor}\binom{2m}{m-10k-5}, \qquad m = 5, 6, 7, \ldots.$$

Indeed, it is easy to see that $A_n = H_{2n}$ and $B_n = H_{2n-1}$. Putting

$$U_m(x) = \sum_{k=0}^{m}\binom{2m}{m-k}x^k,$$

then H_m consists of the subsum of $U_m(x)$ for $x = 1$ corresponding to the summation terms which are congruent to 5 modulo 10. Using the fact that

$$1 + e^{i\pi N/5} + e^{2i\pi N/5} + \cdots + e^{9i\pi N/5}$$

is 0 if $10 \nmid N$ and is 10 otherwise, we get

$$H_m = \frac{1}{10}\sum_{k=0}^{m}\sum_{j=0}^{9}\binom{2m}{m-k}e^{i\pi j(k-5)/5},$$

or, by changing the order of summation,

$$H_m = \frac{1}{10}\sum_{j=0}^{9}(-1)^jU_m(e^{i\pi j/5}).$$

Since also

$$H_m = \frac{1}{10}\sum_{j=0}^{9}(-1)^j U_m(e^{-i\pi j/5}),$$

we get that

$$H_m = \frac{1}{20}\sum_{j=0}^{9}(-1)^j (U_m(e^{i\pi j/5}) + U_m(e^{-i\pi j/5})). \tag{1}$$

Next note that

$$U_m = \sum_{k=0}^{m}\binom{2m}{k}x^{m-k} = \sum_{k=m}^{2m}\binom{2m}{k}x^{k-m},$$

which implies that

$$U_m(x) + U_m(x^{-1}) = \sum_{k=0}^{2m}\binom{2m}{k}x^{m-k} + \binom{2m}{m} = (x+x^{-1})^{2m} + \binom{2m}{m}.$$

In particular,

$$U_m(e^{i\pi j/5}) + U_m(e^{-i\pi j/5}) = (2\cos(\pi j/10))^{2m} + \binom{2m}{m},$$

which together with formula (1) gives

$$H_m = \frac{1}{20}\sum_{j=0}^{9}\left((-1)^j(2\cos(\pi j/10)^{2m} + (-1)^j\binom{2m}{m}\right).$$

The terms involving $\binom{2m}{m}$ cancel. Letting $\theta_j = 2\cos(\pi j/10)$, we have

$$\begin{array}{lllll} \theta_0 = 2, & \theta_1 = \sqrt{\alpha\sqrt{5}}, & \theta_2 = \alpha, & \theta_3 = \sqrt{\alpha^{-1}\sqrt{5}}, & \theta_4 = \alpha^{-1}, \\ \theta_5 = 0, & \theta_6 = -\alpha^{-1}, & \theta_7 = -\sqrt{\alpha^{-1}\sqrt{5}}, & \theta_8 = -\alpha, & \theta_9 = -\sqrt{\alpha\sqrt{5}}. \end{array}$$

Using the above values, we get

$$\begin{aligned} 20H_m &= 2^{2m} - \alpha^m 5^{m/2} + \alpha^{2m} - \alpha^{-m}5^{m/2} + \alpha^{-2m} + \alpha^{-2m} - \alpha^{-m}5^{m/2} + \alpha^{2m} - \alpha^m 5^{m/2} \\ &= 2^{2m} - 2\alpha^m 5^{m/2} - 2(-1)^m\beta^m 5^{m/2} + 2\alpha^{2m} + 2\beta^{2m}, \end{aligned}$$

or

$$H_m = \frac{1}{10}(2^{2m-1} - 5^{m/2}(\alpha^m + (-1)^m\beta^m) + L_{2m}).$$

The desired equalities now follow from Binet's formula. It may be observed that for $n = 1$ and $n = 2$, the sums H_m for $m = 2n$ and $m = 2n - 1$ may be defined as zero, which is also the value obtained when these values are substituted in the closed form expressions.

Also solved by the proposer.

Fibonacci Logarithms

H-652 Proposed by José Luis Díaz-Barrero, Barcelona, Spain (Vol. 45, No. 1, February 2007)

Determine

$$\lim_{n\to\infty}\sum_{k=1}^{n}\ln\left(1+\frac{F_kF_{k+1}}{F_nF_{n+1}}\right)^{F_k/F_{k+1}}.$$

Solution by the editor

Let L be the required limit. We use the fact that

$$F_n=\frac{\alpha^n}{\sqrt{5}}(1+O(\alpha^{-2n})).$$

Let

$$u_n=\sum_{k=1}^{n}\ln\left(1+\frac{F_kF_{k+1}}{F_nF_{n+1}}\right)^{F_k/F_{k+1}}=\sum_{k=1}^{n}\frac{F_k}{F_{k+1}}\ln\left(1+\frac{F_kF_{k+1}}{F_nF_{n+1}}\right).$$

Since $\ln(1+x)<x$ for all $x>0$, we have that

$$0<u_n<\sum_{k=1}^{n}\frac{F_k}{F_{k+1}}\cdot\frac{F_kF_{k+1}}{F_nF_{n+1}}=\frac{1}{F_nF_{n+1}}\sum_{k=1}^{n}F_k^2=1.$$

We now write

$$\begin{aligned}u_n &= \sum_{1\le k<\sqrt{n}}\frac{F_k}{F_{k+1}}\ln\left(1+\frac{F_kF_{k+1}}{F_nF_{n+1}}\right)+\sum_{\sqrt{n}\le k<n}\frac{F_k}{F_{k+1}}\ln\left(1+\frac{F_kF_{k+1}}{F_nF_{n+1}}\right)\\ &:= S_1+S_2.\end{aligned}$$

Obviously,

$$S_1\ll\sum_{k<\sqrt{n}}\frac{F_kF_{k+1}}{F_nF_{n+1}}\ll\frac{\sqrt{n}}{\alpha^{2(n-\sqrt{n})}}=o(1)$$

as $n\to\infty$. If $\sqrt{n}<k\le n$, we then have

$$\begin{aligned}1+\frac{F_kF_{k+1}}{F_nF_{n+1}} &= 1+\frac{\alpha^{2k+1}+O(1)}{\alpha^{2n+1}(1+O(\alpha^{-2n}))}\\ &= 1+\alpha^{2(k-n)}+O\left(\alpha^{-2n}\right)\\ &= (1+\alpha^{2(k-n)})(1+O(\alpha^{-2n})),\end{aligned}$$

and

$$\frac{F_k}{F_{k+1}}=\alpha^{-1}(1+O(\alpha^{-2k}))=\alpha^{-1}(1+O(\alpha^{-\sqrt{n}})),$$

therefore

$$\begin{aligned} S_2 &= \frac{1}{\alpha} \sum_{\sqrt{n}<k\le n} (1+O(\alpha^{-\sqrt{n}})) \left(\ln(1+\alpha^{-2(n-k)}) + \ln(1+O(\alpha^{-2n}))\right) \\ &= \frac{1}{\alpha} \sum_{\sqrt{n}<k\le n} \ln(1+\alpha^{-2(n-k)}) + O(n\alpha^{-\sqrt{n}}) \\ &= \frac{1}{\alpha} \left(\sum_{k=1}^{n} \ln(1+\alpha^{-2(n-k)}) - S_3 \right) + o(1), \end{aligned}$$

where

$$S_3 = \sum_{1\le k<\sqrt{n}} \ln(1+\alpha^{-2(n-k)}).$$

Since clearly

$$S_3 < \frac{\sqrt{n}}{\alpha^{2(n-\sqrt{n})}} = o(1),$$

we get that if we write

$$v_n = \sum_{k=1}^{n} \ln(1+\alpha^{-2(n-k)}),$$

then the required limit is $\alpha^{-1}\lim_{n\to\infty} v_n$. But clearly,

$$v_n = \sum_{k=0}^{n-1} \ln(1+\alpha^{-2k}),$$

and therefore the limit of v_n is simply the sum of the series

$$\sum_{k=0}^{\infty} \ln(1+\alpha^{-2k}).$$

To compute it with any precision, we note that its tail at N is at most

$$\sum_{k\ge N} \ln(1+\alpha^{-2k}) < \sum_{k\ge N} \alpha^{-2k} = \alpha^{-2N}\frac{1}{1-\alpha^{-2}} = \alpha^{-2N+1},$$

so

$$\left| L - \frac{1}{\alpha}\sum_{k=0}^{N-1} \ln(1+\alpha^{-2k}) \right| < \alpha^{-2N}.$$

Using the above with $N = 10$, we get that the first three decimals of L are 0.767. It is not clear if this expressions admits a closed form.

Also solved by Paul S. Bruckman, H.-J. Seiffert and the proposer.

Harmonic Numbers, $\zeta(2)$ and $\zeta(3)$

H-653 Proposed by Ovidiu Furdui, Kalamazoo, MI (Vol. 45, No. 1, February 2007)

Let $n \geq 3$ be a natural number. Prove the following formula

$$\sum_{k=1}^{\infty} \frac{H_k}{k^2(k+1)(k+2)\cdots(k+n)} = \frac{2}{n!}\left(\zeta(3) - \frac{\pi^2}{8} + \frac{1}{4}\right) - \frac{1}{n!}\sum_{k=3}^{n}\frac{1}{k}\left(\frac{\pi^2}{6} - \sum_{j=1}^{k-1}\frac{1}{j^2}\right),$$

where $H_k = \sum_{j=1}^{k} 1/j$ is the kth harmonic number and $\zeta(3) = \sum_{j=1}^{\infty} 1/j^3$ is the celebrated Apéry constant.

Solution by H.-J. Seiffert, Berlin, Germany

If $P_{k,n} = k(k+1)\cdots(k+n),\ k \geq 1,\ n \geq 0$, then

$$\frac{1}{P_{k,n}} - \frac{1}{P_{k+1,n}} = \frac{n+1}{P_{k,n+1}}. \tag{2}$$

We consider the series

$$S_n = \sum_{k=1}^{\infty} \frac{H_k}{kP_{k,n}}, \qquad T_n = \sum_{k=1}^{\infty} \frac{H_k}{P_{k,n}}, \qquad U_n = \sum_{k=1}^{\infty} \frac{1}{kP_{k,n}}, \qquad V_n = \sum_{k=1}^{\infty} \frac{1}{P_{k,n}}.$$

The series S_n and U_n converge for $n \geq 0$ while T_n and V_n converge for $n \geq 1$. Multiplying (2) by H_k/k and summing over all $k \geq 1$ gives

$$S_n - T_{n+1} = (n+1)S_{n+1}, \qquad n \geq 0. \tag{3}$$

Above we used the fact that $kP_{k+1,n} = P_{k,n+1}$. Since $H_{k+1} = H_k + 1/(k+1)$, by (2), we get

$$\frac{H_k}{P_{k,n}} - \frac{H_{k+1}}{P_{k+1,n}} = \frac{(n+1)H_k}{P_{k,n+1}} - \frac{1}{(k+1)P_{k+1,n}}, \qquad k \geq 1,\ n \geq 1.$$

Summing over all $k \geq 1$ and noting the telescoping on the left hand side, one almost immediately obtains

$$T_{n+1} = \frac{U_n}{n+1}, \qquad n \geq 0. \tag{4}$$

Multiplying (2) by $1/k$ and then summing over all $k \geq 1$ yields

$$U_n - V_{n+1} = (n+1)U_{n+1}, \qquad n \geq 0. \tag{5}$$

Finally, summing (2) over all $k \geq 1$, one easily gets

$$V_{n+1} = \frac{1}{(n+1)(n+1)!}, \qquad n \geq 0. \tag{6}$$

Based on (5) and (6), a simple induction argument shows that

$$U_n = \frac{1}{n!}\left(U_0 - \sum_{j=1}^{n}\frac{1}{j^2}\right), \qquad n \geq 0; \tag{7}$$

empty sums are understood to be zero. From (3), (4) and (7), it follows that

$$S_{n+1} = \frac{S_n}{n+1} - \frac{1}{(n+1)(n+1)!}\left(U_0 - \sum_{j=1}^{n}\frac{1}{j^2}\right), \qquad n \geq 0. \tag{8}$$

Using Euler's results $U_0 = \pi^2/6$, as well as $S_0 = 2\zeta(3)$, we find from (8) that $S_1 = 2\zeta(3) - \pi^2/6$. Now, based on (8), a simple induction argument shows that

$$S_n = \frac{2}{n!}\left(\zeta(3) - \frac{\pi^2}{8} + \frac{1}{4}\right) - \frac{1}{n!}\sum_{k=3}^{n}\frac{1}{k}\left(\frac{\pi^2}{6} - \sum_{j=1}^{k-1}\frac{1}{j^2}\right), \qquad n \geq 2.$$

This proves the required identity.

Editor's comment. In the original statement of the problem, the last sum on the right hand side was erroneously written as '$\sum_{j=1}^{n-1} 1/j^2$' instead of '$\sum_{j=1}^{k-1} 1/j^2$'. The editor apologizes for this oversight.

Also solved by K. N. Boyadzhiev, Paul S. Bruckman and the proposer.

Late Acknowledgement. H-648 and H-649 were also solved by Paul S. Bruckman.

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