ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
FLORIAN LUCA

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PROBLEMS PROPOSED IN THIS ISSUE

H-735 Proposed by Paul S. Bruckman, BC

Let $F_m(x) = \sum_{n=0}^{\infty} \binom{2n+m}{n} x^n$, where m is any real number and $|x| < 1/4$. Also let $\theta(x) = (1-4x)^{1/2}$. For brevity, write $F_m = F_m(x)$, $\theta = \theta(x)$. Prove the following:

(a) $F_0 = \frac{1}{\theta}$, $F_1 = \frac{(1-\theta)}{2x\theta}$;

(b) for all real m, $\frac{F_m}{F_0} = \left(\frac{F_1}{F_0}\right)^m$;

(c) for all real m, $\sum_{k=0}^{n} \binom{2k+m}{k}\binom{2n-2k-m}{n-k} = 4^n, \quad n = 0, 1, 2, \ldots$.

H-736 Proposed by Hideyuki Ohtsuka, Saitama, Japan

The Tribonacci numbers T_n satisfy $T_0 = 0$, $T_1 = T_2 = 1$, $T_{n+3} = T_{n+2} + T_{n+1} + T_n$ for $n \geq 0$. Find an explicit formula for the sum $\sum_{k=1}^{n} T_k^3$.

H-737 Proposed by Hideyuki Ohtsuka, Saitama, Japan

Let $\binom{n}{k}_F$ denote the Fibonomial coefficient. For an odd prime p and a positive integer n, prove that

$$\binom{np-1}{p-1}_F \equiv (-1)^{\frac{(n-1)(p-1)}{2}} \pmod{F_p^2 L_p}.$$

H-738 Proposed by H. Ohtsuka, Saitama, Japan

Let $\binom{n}{k}_F$ denote the Fibonomial coefficient. For $n \geq 1$, prove that

(i) $\sum_{k=0}^{2n-1} L_k^2 \binom{2n-1}{k}_F^2 = \frac{L_{4n-1}+1}{L_{4n-1}-1} \sum_{k=0}^{2n} \binom{2n}{k}_F^2,$

$$\text{(ii)} \quad \sum_{\substack{a+b=2n \\ a,b>0}} L_a L_b \binom{2n-1}{a}_F \binom{2n-1}{b}_F = \frac{L_{4n-1}-3}{L_{4n-1}-1} \sum_{k=0}^{2n} \binom{2n}{k}_F^2 .$$

SOLUTIONS

Summatory Function of the Riemann Zeta Function

H-709 Proposed by Ovidiu Furdui, Campia Turzii, Romania (Vol. 49, No. 4, November 2011)

a) Let a be a positive real number. Calculate,

$$\lim_{n\to\infty} a^n \left(n - \zeta(2) - \zeta(3) - \cdots - \zeta(n)\right),$$

where ζ is the Riemann zeta function.

b) Let a be a real number such that $|a| < 2$. Prove that,

$$\sum_{n=2}^{\infty} a^n \left(n - \zeta(2) - \zeta(3) - \cdots - \zeta(n)\right) = a\left(\frac{\Psi(2-a)+\gamma}{1-a} - 1\right),$$

where Ψ denotes the Digamma function.

Solution by the proposer.

We need the following lemma.

Lemma 1. *The following limit holds* $\lim_{x\to\infty} 2^x \left(\zeta(x) - 1\right) = 1$.

Proof. We note that if $x > 1$ we have $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} = 1 + \frac{1}{2^x} + \sum_{n=3}^{\infty} \frac{1}{n^x} > 1 + \frac{1}{2^x}$, and hence,

$$1 < 2^x \left(\zeta(x) - 1\right). \tag{1}$$

On the other hand, $\sum_{n=3}^{\infty} \frac{1}{n^x} < \int_2^{\infty} \frac{1}{t^x} dt = \frac{2^{1-x}}{x-1}$, from which it follows that

$$2^x \left(\zeta(x) - 1\right) < \frac{x+1}{x-1}. \tag{2}$$

From (1) and (2) we get that $\lim_{x\to\infty} 2^x \left(\zeta(x) - 1\right) = 1$. □

Now we are ready to solve the problem. First, we prove that

$$S_n = \sum_{k=1}^{\infty} \frac{1}{k(k+1)^n} = n - \zeta(2) - \zeta(3) - \cdots - \zeta(n).$$

We have, since

$$\frac{1}{k(k+1)^n} = \frac{1}{k(k+1)^{n-1}} - \frac{1}{(k+1)^n},$$

that

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)^n} = \sum_{k=1}^{\infty} \frac{1}{k(k+1)^{n-1}} - \sum_{k=1}^{\infty} \frac{1}{(k+1)^n},$$

and hence, $S_n = S_{n-1} - (\zeta(n) - 1)$. Iterating this equality we obtain

$$S_n = S_1 - (\zeta(2) + \zeta(3) + \cdots + \zeta(n) - (n-1))$$

and, since $S_1 = \sum_{k=1}^{\infty} 1/(k(k+1)) = 1$, we obtain $S_n = n - \zeta(2) - \zeta(3) - \cdots - \zeta(n)$.

a) We prove that $\lim_{n\to\infty} a^n (n - \zeta(2) - \zeta(3) - \cdots - \zeta(n))$ equals 1 for $a = 2$, 0 for $a \in (0, 2)$, and ∞ when $a > 2$. First we consider the case when $a = 2$. Let $L = \lim_{n\to\infty} 2^n S_n$. Since S_n verifies the recurrence formula $S_n = S_{n-1} - (\zeta(n) - 1)$, it follows that

$$2^n S_n = 2 \cdot 2^{n-1} S_{n-1} - 2^n(\zeta(n) - 1).$$

Letting n tend to ∞ in the preceding equality and using the lemma we get $L = 2L - 1$, from which it follows that $L = 1$. If $a < 2$, we have that $L = \lim_{n\to\infty} a^n S_n = \lim_{n\to\infty} 2^n S_n \cdot \lim_{n\to\infty}(a/2)^n = 0$, and if $a > 2$ we get, based on the same reasoning, that $L = \infty$.

b) Clearly when $a = 0$ there is nothing to prove so we consider the case when $a \neq 0$. We have,

$$\begin{aligned}\sum_{n=2}^{\infty} a^n (n - \zeta(2) - \zeta(3) - \cdots - \zeta(n)) &= \sum_{n=2}^{\infty} a^n \sum_{k=1}^{\infty} \frac{1}{k(k+1)^n} = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=2}^{\infty} \left(\frac{a}{k+1}\right)^n \\ &= a^2 \sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+1-a)}.\end{aligned}$$

We distinguish two cases here.

Case $a = 1$. In this case we have, based on the preceding calculations,

$$\sum_{n=2}^{\infty} (n - \zeta(2) - \zeta(3) - \cdots - \zeta(n)) = \sum_{k=1}^{\infty} \frac{1}{k^2(k+1)} = \zeta(2) - 1.$$

Case $a \neq 1$. We have,

$$\begin{aligned}\sum_{n=2}^{\infty} a^n (n - \zeta(2) - \zeta(3) - \cdots - \zeta(n)) &= a^2 \sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+1-a)} \\ &= a^2 \sum_{k=1}^{\infty} \left(\frac{1}{k(k+1-a)} - \frac{1}{(k+1)(k+1-a)}\right) \\ &= a^2 \left(\sum_{k=1}^{\infty} \frac{1}{1-a}\left(\frac{1}{k} - \frac{1}{k+1-a}\right) - \sum_{k=1}^{\infty} \frac{1}{a}\left(\frac{1}{k+1-a} - \frac{1}{k+1}\right)\right) \\ &= a^2 \left(\sum_{k=1}^{\infty} \frac{1}{1-a}\left(\frac{1}{k} - \frac{1}{k+1-a}\right) - \sum_{k=1}^{\infty} \frac{1}{a}\left(\frac{1}{k+1-a} - \frac{1}{k} + \frac{1}{k} - \frac{1}{k+1}\right)\right) \\ &= a^2 \left(\frac{1}{a(1-a)} \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1-a}\right) - \frac{1}{a} \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)\right) \\ &= a\left(\frac{\Psi(2-a) + \gamma}{1-a} - 1\right),\end{aligned}$$

and the problem is solved.

Remark. It is worth mentioning that this formula is consistent with the case when $a = 1$, since

$$\lim_{a\to 1} a\left(\frac{\Psi(2-a)+\gamma}{1-a}-1\right) = \Psi'(1)-1 = \frac{\pi^2}{6}-1.$$

Also solved by Khristo Boyadzhiev, Paul S. Bruckman, and Anastasios Kotronis.

A Double Generating Function for Ternary Words

H-710 Proposed by Emeric Deutsch, Polytechnic Institute of NYU, Brooklyn, NY
(Vol. 49, No. 4, November 2011)

Let $a_{n,k}$ denote the number of ternary words (i.e., finite sequences of 0's, 1's and 2's) of length n and having k occurrences of 01's. Find the generating function $G(t,z) = \sum_{k\ge 0, n\ge 0} a_{n,k} t^k z^n$.

Solution by Helmut Prodinger, Stellenbosch, South Africa.

Such questions can be answered in a fairly automatic fashion using a finite automaton. Set

$$A := \begin{bmatrix} 2z & z \\ zt+z & z \end{bmatrix},$$

then

$$G(t,z) = \begin{bmatrix} 1 & 0 \end{bmatrix} (I-A)^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{1-3z+z^2-z^2t}.$$

Also solved by Paul S. Bruckman and the proposer.

Inequalities With Square Roots of Fibonacci Numbers

H-711 Proposed by Hideyuki Ohtsuka, Saitama, Japan
(Vol. 49, No. 4, November 2011)

Let $R_n = \sqrt{F_{n+3}} + \sqrt{F_{n+4}}$ for $n \ge 0$. Prove that

$$R_n - R_2 + 2 \le \sum_{k=1}^{n} \sqrt{F_k} \le R_n - R_1 + 1.$$

Solution by the proposer.

First, we prove the following lemma.

Lemma. For positive integer n,

(1) $\sqrt{F_{2n+1}} + \sqrt{F_{2n+3}} - \sqrt{F_{2n+5}} > 0,$
(2) $\sqrt{F_{2n+1}} + \sqrt{F_{2n+2}} + \sqrt{F_{2n+3}} + \sqrt{F_{2n+4}} - \sqrt{F_{2n+5}} - \sqrt{F_{2n+6}} > 0,$
(3) $\sqrt{F_{2n}} + \sqrt{F_{2n+1}} + \sqrt{F_{2n+2}} + \sqrt{F_{2n+3}} - \sqrt{F_{2n+4}} - \sqrt{F_{2n+5}} < 0.$

Proof. For (1), note that

$$(\sqrt{F_{2n+1}} + \sqrt{F_{2n+3}})^2 - (\sqrt{F_{2n+5}})^2 = 2\sqrt{F_{2n+2}^2+1} - 2F_{2n+2} > 0. \qquad (3)$$

Therefore, $\sqrt{F_{2n+1}} + \sqrt{F_{2n+3}} - \sqrt{F_{2n+5}} > 0$.

For (2), we consider the following A_n:

$$\begin{aligned}A_n &= \left(\sqrt{F_{2n+6}}+\sqrt{F_{2n+4}}+\sqrt{F_{2n+2}}\right)\\ &\quad \left\{\left(\sqrt{F_{2n+3}}+\sqrt{F_{2n+1}}-\sqrt{F_{2n+5}}\right)-\left(\sqrt{F_{2n+6}}-\sqrt{F_{2n+4}}-\sqrt{F_{2n+2}}\right)\right\}\\ &> \left(\sqrt{F_{2n+5}}+\sqrt{F_{2n+3}}+\sqrt{F_{2n+1}}\right)\left(\sqrt{F_{2n+3}}+\sqrt{F_{2n+1}}-\sqrt{F_{2n+5}}\right)\\ &- \left(\sqrt{F_{2n+6}}+\sqrt{F_{2n+4}}+\sqrt{F_{2n+2}}\right)\left(\sqrt{F_{2n+6}}-\sqrt{F_{2n+4}}-\sqrt{F_{2n+2}}\right)\\ &= 2\left(\sqrt{F_{2n+3}^2-1}+\sqrt{F_{2n+2}^2+1}-F_{2n+4}\right).\end{aligned}$$

Here

$$\begin{aligned}&\left(\sqrt{F_{2n+3}^2-1}+\sqrt{F_{2n+2}^2+1}\right)^2-F_{2n+4}^2\\ &\qquad = 2\sqrt{(F_{2n+2}F_{2n+3})^2+F_{2n+3}^2-F_{2n+2}^2-1}-2F_{2n+2}F_{2n+3}>0.\end{aligned}$$

Therefore, $\sqrt{F_{2n+3}^2-1}+\sqrt{F_{2n+2}^2+1}-F_{2n+4}>0$. Hence, $A_n>0$. We have

$$\left(\sqrt{F_{2n+3}}+\sqrt{F_{2n+1}}-\sqrt{F_{2n+5}}\right)-\left(\sqrt{F_{2n+6}}-\sqrt{F_{2n+4}}-\sqrt{F_{2n+2}}\right)>0.$$

Thus, we obtain (2). Part (3) can be obtained similarly. □

We define S_n as follows. For positive integer n, $S_n=\sum_{k=1}^n\sqrt{F_k}-\sqrt{F_{n+3}}-\sqrt{F_{n+4}}$. For positive integer m, we show

(i) $S_{2m+1}>S_{2m}$, (ii) $S_{2m+2}>S_{2m}$ and (iii) $S_{2m+1}<S_{2m-1}$.

(i) By Lemma (1),

$$S_{2m+1}-S_{2m}=\sqrt{F_{2m+1}}+\sqrt{F_{2m+3}}-\sqrt{F_{2m+5}}>0.$$

(ii) By Lemma (2),

$$S_{2m+2}-S_{2m}=\sqrt{F_{2m+1}}+\sqrt{F_{2m+2}}+\sqrt{F_{2m+3}}+\sqrt{F_{2m+4}}-\sqrt{F_{2m+5}}-\sqrt{F_{2m+6}}>0.$$

(iii) By Lemma (3),

$$S_{2m+1}-S_{2m-1}=\sqrt{F_{2m}}+\sqrt{F_{2m+1}}+\sqrt{F_{2m+2}}+\sqrt{F_{2m+3}}-\sqrt{F_{2m+4}}-\sqrt{F_{2m+5}}<0.$$

By (i), (ii), and (iii) we have

$$S_2<S_4<S_6<\cdots<S_5<S_3<S_1.$$

Thus, $S_2\le S_n\le S_1$. Putting $R_n=\sqrt{F_{n+3}}+\sqrt{F_{n+4}}$, then we have

$$2-R_2\le\sum_{k=1}^n\sqrt{F_k}-R_n\le 1-R_1.$$

Therefore,

$$R_n-R_2+2\le\sum_{k=1}^n\sqrt{F_k}\le R_n-R_1+1.$$

Also solved by Paul S. Bruckman, Kenneth B. Davenport, and Zbigniew Jakubczyk.

Convolutions With Middle Binomial Coefficients

H-712 Proposed by N. Gauthier, Royal Military College of Canada, Kingston, ON
(Vol. 50, No. 1, February 2012)

The nth central binomial coefficient is, for an integer $n \geq 0$: $B_n = \binom{2n}{n}$. Then, for a nonnegative integer m, define the convolution

$$b_m(n) = \sum_{k=0}^{n} k^m B_{n-k} B_k,$$

where $b_0(n) = \sum_{k=0}^{n} B_{n-k} B_k$. Prove the following recurrence,

$$b_m(n) = \frac{2^{2n-m}(2m-1)!!(n)_m}{m!} - \sum_{k=1}^{m-1} S_m^{(k)} b_k(n).$$

In this expression, the sum on the right-hand side is taken to vanish when $m = 0, 1$, and the coefficients are Stirling numbers of the first kind, $\{S_m^{(k)} : 1 \leq k \leq m\}$. Also,

$$(2m-1)!! = 1 \cdot 3 \cdot 5 \cdots (2m-1); \qquad (n)_m = n(n-1)\ldots(n-m+1),$$

where, by convention, $(2m-1)!! = 1$ and $(n)_m = 1$ for $m = 0$.

Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY.

We shall first derive another recurrence relation for $b_m(n)$. Let $B_m(x) = \sum_{n=0}^{\infty} b_m(n) x^n$, and define $C_m(x) = \sum_{k=0}^{\infty} k^m B_k x^k$. We have

$$C_0(x) = \frac{d}{dx}\left(\sum_{k=0}^{\infty} \frac{1}{k+1}\binom{2k}{k} x^{k+1}\right) = \frac{d}{dx}\left(\frac{1-\sqrt{1-4x}}{2}\right) = (1-4x)^{-1/2}.$$

The convolution in the definition of $b_m(n)$ leads to $B_m(x) = C_0(x)C_m(x) = (1-4x)^{-1/2}C_m(x)$. In particular,

$$B_0(x) = (1-4x)^{-1} = \sum_{n=0}^{\infty} 4^n x^n,$$

hence, $b_0(n) = 4^n$. For $m \geq 1$, from $C_{m-1}(x) = (1-4x)^{1/2} B_{m-1}(x)$, we obtain

$$C_m(x) = xC'_{m-1}(x) = x(1-4x)^{1/2} B'_{m-1}(x) - 2x(1-4x)^{-1/2} B_{m-1}(x).$$

Thus,

$$\begin{aligned} B_m(x) &= (1-4x)^{-1/2} C_m(x) \\ &= xB'_{m-1}(x) - 2x(1-4x)^{-1} B_{m-1}(x) \\ &= \sum_{n=0}^{m} n b_{m-1}(n) x^n - 2x \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} 4^{n-k} b_{m-1}(k)\right) x^n. \end{aligned}$$

Comparison of the coefficient of x^n yields the recurrence

$$b_m(n) = n b_{m-1}(n) - 2\sum_{k=0}^{n-1} 4^{n-1-k} b_{m-1}(k), \qquad m \geq 1. \tag{4}$$

Since $b_0(n) = 4^n$, the identity stated in the problem holds when $m = 0$. To finish the proof, we use induction on $m \geq 1$ to prove the equivalent form

$$\sum_{k=1}^{m} S_m^{(k)} b_k(n) = 4^{n-m}(2m)_m \binom{n}{m}. \tag{5}$$

It is easy to verify that (5) is true when $m = 1$ because the recurrence (4) and $b_0(n) = 4^n$ together imply that $b_1(n) = 2n \cdot 4^{n-1}$. Assuming that (5) is true for some $m \geq 1$, it remains to show that it is also valid when m is replaced by $m+1$. Using a well-known recurrence that $S_m^{(k)}$ satisfies, the fact that $S_m^{(0)} = S_m^{(m+1)} = 0$, the recurrence (4), the induction hypothesis (5), the "hockey-stick" theorem, and simple algebra, we find

$$\begin{aligned}
\sum_{k=1}^{m+1} S_{m+1}^{(k)} b_k(n) &= \sum_{k=1}^{m+1} \left(S_m^{(k-1)} - m S_m^{(k)} \right) b_k(n) \\
&= \sum_{k=1}^{m} S_m^{(k)} b_{k+1}(n) - m \sum_{k=1}^{m} S_m^{(k)} b_k(n) \\
&= \sum_{k=1}^{m} S_m^{(k)} \left(n b_k(n) - 2 \sum_{j=0}^{n-1} 4^{n-1-j} b_k(j) \right) - m \sum_{k=1}^{m} S_m^{(k)} b_k(n) \\
&= (n-m) \sum_{k=1}^{m} S_m^{(k)} b_k(n) - 2 \sum_{j=0}^{n-1} 4^{n-1-j} \sum_{k=1}^{m} S_m^{(k)} b_k(j) \\
&= (n-m) \cdot 4^{n-m}(2m)_m \binom{n}{m} - 2 \cdot 4^{n-m-1}(2m)_m \sum_{j=1}^{n-1} \binom{j}{m} \\
&= (m+1) \cdot 4^{n-m}(2m)_m \binom{n}{m+1} - 2 \cdot 4^{n-m-1}(2m)_m \binom{n}{m+1} \\
&= 4^{n-m-1}(4m+2)(2m)_m \binom{n}{m+1} \\
&= 4^{n-m-1}(2m+2)_{m+1} \binom{n}{m+1},
\end{aligned}$$

thereby completing the induction.

Also solved by Paul Bruckman, Andrew Gibson, Matthew Roberson & Cecil Rousseau, and the proposer.

Errata. Kenneth B. Davenport pointed out that Problem **H-730** is exactly the same as **H-720**. The editor apologizes for the oversight.