## ADVANCED PROBLEMS AND SOLUTIONS

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### PROBLEMS PROPOSED IN THIS ISSUE

**H-769 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.**

Prove that the inequality

$$\frac{F_1^6}{(F_1^4+F_1^2F_2^2+F_2^4)(\sqrt{2}F_1+F_2)}+\frac{F_2^6}{(F_2^4+F_2^2F_3^2+F_3^4)(\sqrt{2}F_2+F_3)}+\cdots$$
$$+\frac{F_{n-1}^6}{(F_{n-1}^4+F_{n-1}^2F_n^2+F_n^4)(\sqrt{2}F_{n-1}+F_n)}+\frac{F_n^6}{(F_n^4+F_n^2F_1^2+F_1^4)(\sqrt{2}F_n+F_1)}$$
$$\geq\frac{\sqrt{2}-1}{3}(F_{n+2}-1)$$

holds for all positive integers $n$.

**H-770 Proposed by H. Ohtsuka, Saitama, Japan.**

For an integer $n \geq 0$, find a closed form expression for the sum

$$S(n) := \sum_{k=0}^{n}\frac{1}{(L_{2^{k+1}}+1)(L_{2^k}+c)(L_{2^{k+1}}+c)\cdots(L_{2^n}+c)},$$

where $c \neq -L_{2^k}$ for $0 \leq k \leq n$.

**H-771 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.**

Let $m > 0$ and $\Gamma : (0,\infty) \to (0,\infty)$ be the gamma function. Calculate

$$\lim_{n\to\infty}\int_{\sqrt[n]{n!}}^{\sqrt[n+1]{(n+1)!}}\Gamma\left(\frac{x}{n}\sqrt[n]{F_n^m}\right)dx.$$

**H-772** **Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.**

If $ABC$ is a nonisosceles triangle then prove that

$$\sum_{\substack{cyclic\\permutations}} \frac{a^8}{(bF_n^2 + cF_{n+1}^2)(a-b)^2(a-c)^2} > \frac{288r^3\sqrt{3}}{F_{2n+1}}.$$

Here, $a, b, c, r$ are the lengths of the sides and the radius of the inscribed circle of the triangle $ABC$, respectively.

## SOLUTIONS

### On Sums of Squares of Fibonomial Coefficients

**H-738** **Proposed by H. Ohtsuka, Saitama, Japan.**
**(Vol. 51, No. 2, May 2013)**

Let $\binom{n}{k}_F$ denote the Fibonomial coefficient. For $n \geq 1$, prove that

(i) $\displaystyle\sum_{k=0}^{2n-1} L_k^2 \binom{2n-1}{k}_F^2 = \frac{L_{4n-1}+1}{L_{4n-1}-1} \sum_{k=0}^{2n} \binom{2n}{k}_F^2,$

(ii) $\displaystyle\sum_{\substack{a+b=2n\\a,b>0}} L_a L_b \binom{2n-1}{a}_F \binom{2n-1}{b}_F = \frac{L_{4n-1}-3}{L_{4n-1}-1} \sum_{k=0}^{2n} \binom{2n}{k}_F^2.$

**Solution by the proposer**

(i) The following identities are known (see [2]):

(A) $F_{2n} = F_n L_n$,
(B) $L_n^2 = 5F_n^2 + 4(-1)^n$,
(C) $L_n L_m = L_{n+m} + (-1)^m L_{n-m}$,
(D) $5F_n F_m = L_{n+m} - (-1)^m L_{n-m}$.

We use two properties

$$\binom{n}{k}_F = \frac{F_n}{F_k}\binom{n-1}{k-1}_F \quad \text{and} \quad \binom{n}{k}_F = \binom{n}{n-k}_F. \tag{1}$$

By (1) above, we have

$$\sum_{k=0}^{m} F_k^2 \binom{m}{k}_F^2 = \sum_{k=1}^{m} F_m^2 \binom{m-1}{k-1}_F^2 = F_m^2 \sum_{k=0}^{m-1} \binom{m-1}{k}_F^2. \tag{2}$$

For odd $m$ we have

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k}_F^2 = 0, \tag{3}$$

because

$$LHS = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{m-k}^2 = -\sum_{k=0}^{m} (-1)^k \binom{m}{k}_F^2 = -LHS.$$

It was shown in [1] that

$$\sum_{k=0}^{2n} \binom{2n}{k}_F^2 = \prod_{k=1}^{n} \frac{L_{2k}F_{2(2k-1)}}{F_{2k}}. \tag{4}$$

We have

$$\begin{aligned}
\sum_{k=0}^{2n-1} L_k^2 \binom{2n-1}{k}_F^2 &= 5\sum_{k=0}^{2n-1} F_k^2 \binom{2n-1}{k}_F^2 + 4\sum_{k=0}^{2n-1} (-1)^k \binom{2n-1}{k}_F^2 \qquad \text{(by } (B))\\
&= 5\sum_{k=0}^{2n-1} F_k^2 \binom{2n-1}{k}_F \qquad \text{(by (3))}\\
&= 5F_{2n-1}^2 \sum_{k=0}^{2n-2} \binom{2n-2}{k}_F^2 \qquad \text{(by (2))}\\
&= \frac{5F_{2n-1}^2 F_{2n}}{L_{2n}F_{2(2n-1)}} \sum_{k=0}^{2n} \binom{2n}{k}_F^2 \qquad \text{(by (4))}\\
&= \frac{5F_{2n}F_{2n-1}}{L_{2n}L_{2n-1}} \sum_{k=0}^{2n} \binom{2n}{k}_F^2 \qquad \text{(by } (A))\\
&= \frac{L_{4n-1}+1}{L_{4n-1}-1} \sum_{k=0}^{2n} \binom{2n}{k}_F^2 \qquad \text{(by } (C) \text{ and } (D)).
\end{aligned}$$

(ii) The following identity is known (see [3]):

$$2\binom{m}{k}_F = L_k \binom{m-1}{k}_F + L_{m-k}\binom{m-1}{k-1}_F.$$

Squaring both sides of the above identity, we have

$$4\binom{m}{k}_F^2 = L_k^2\binom{m-1}{k}_F^2 + L_{m-k}^2\binom{m}{k-1}_F^2 + 2L_kL_{m-k}\binom{m-1}{k}_F\binom{m-1}{k-1}_F.$$

Using this identity, we have

$$\begin{aligned}
4\sum_{k=1}^{m-1}\binom{m}{k}_F^2 &= \sum_{k=1}^{m-1}\left(L_k^2\binom{m-1}{k}_F^2 + L_{m-k}^2\binom{m-1}{m-k}_F^2 + 2L_kL_{m-k}\binom{m-1}{k}_F\binom{m-1}{k-1}_F\right)\\
&= 2\sum_{k=1}^{m-1} L_k^2\binom{m-1}{k}_F^2 + 2\sum_{\substack{a+b=m\\ a,b>0}} L_aL_b\binom{m-1}{a}_F\binom{m-1}{b}_F.
\end{aligned}$$

Therefore, putting $m = 2n$, we have

$$\sum_{\substack{a+b=2n\\a,b>0}} L_a L_b \binom{2n-1}{a}_F \binom{2n-1}{b}_F = 2\sum_{k=1}^{2n-1}\binom{2n}{k}_F^2 - \sum_{k=1}^{2n-1} L_k^2 \binom{2n-1}{k}_F^2$$

$$= 2\left(\sum_{k=0}^{2n}\binom{2n}{k}_F^2 - 2\right) - \left(\sum_{k=0}^{2n-1} L_k^2\binom{2n-1}{k}_F^2 - 4\right)$$

$$= 2\sum_{k=0}^{2n}\binom{2n}{k}_F^2 - \frac{L_{4n-1}+1}{L_{4n-1}-1}\sum_{k=0}^{2n}\binom{2n}{k}^2 \qquad \text{(by } (i))$$

$$= \frac{L_{4n-1}-3}{L_{4n-1}-1}\sum_{k=0}^{2n}\binom{2n}{k}_F^2.$$

## References


[1] E. Kılıç, I. Akkuş and H. Ohtsuka, *Some generalized Fibonomial sums related to the Gaussian q-binomial sums*, Bull. Math. Sci. Math. Roumanie Tome, **55.103** (2012), 51–61.

[2] S. Vajda, *Fibonacci and Lucas Numbers and the Golden Section*, DOVER, 2008.

[3] E. W. Weisstein, *Fibonomial Coefficient*, From MathWorld-A Wolfram Web Resource `http://mathworld.wolfram.com/FibonomialCoefficient.html`.


## More Sums of Squares of Fibonomial Coefficients

**H-739 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 51, No. 3, August 2013)**

Define the generalized Fibonomial coefficient $\binom{n}{k}_{F;m}$ by

$$\binom{n}{k}_{F;m} = \prod_{j=1}^{k}\frac{F_{m(n-j+1)}}{F_{mj}} \quad (\text{for } n \geq k > 0) \quad \text{with} \quad \binom{n}{0}_{F;m} = 1.$$

Prove that

$$\sum_{k=0}^{n}\alpha^{2mn(n-2k)}\binom{n}{k}_{F;2m}^2 = \sum_{k=0}^{2n}(-1)^{(m+1)(n-k)}\binom{2n}{k}_{F;m}^2.$$

**Solution by E. Kılıç and I. Akkuş, Turkey.**

Our way is to mechanically compute the desired sums by the qZeilberger algorithm (qZeilberger's own version, which is a Mathematica program).

The Gaussian $q$-binomial coefficient $\binom{n}{m}_q$ is defined, for all real $n$ and integers $m$ with $m \geq 0$,

$$\binom{n}{k}_{q^m} := \frac{(q^m;q^m)_n}{(q^m;q^m)_k(q^m;q^m)_{n-k}}$$

and as zero otherwise, where

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

The Binet form is

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1}\frac{1-q^n}{1-q}$$

with $q = \beta/\alpha = -\alpha^{-2}$. The link between the Fibonomial and Gaussian $q$-binomial coefficients is

$$\binom{n}{k}_{F;m} = \alpha^{mk(n-k)}\binom{n}{k}_{q^m} \quad \text{with} \quad q = -\alpha^{-2}.$$

The claimed identity is

$$\sum_{k=0}^{n} \alpha^{2mn(n-2k)}\binom{n}{k}^2_{F;2m} = \sum_{k=0}^{2n}(-1)^{(m+1)(n-k)}\binom{2n}{k}^2_{F;m}.$$

If we convert the claimed identity into $q$-notation, then we rewrite its LHS and RHS in terms of $q$-binomials as

$$(-1)^{mn}q^{-mn^2}\sum_{k=0}^{n} q^{2mk^2}\binom{n}{k}^2_{q^{2m}},$$

and

$$(-1)^{n(m+1)}\sum_{k=0}^{2n}(-1)^k q^{mk(k-2n)}\binom{2n}{k}^2_{q^m},$$

respectively.

The algorithm gives us the recurrence relation for both LHS and RHS :

$$\texttt{SUM[n]=}\frac{(-1)^m q^{m(1-2n)}(1+q^{2mn})(1-q^{2m(2n-1)})}{1-q^{2nm}}\texttt{SUM[n - 1]} \ .$$

Since LHS(0)=RHS(0)= 1, we get that they are equal.

**Also solved by the proposer**.

## Counting Dominating Sets in Paths

**H-740 Proposed by Saeid Alikhani, Yazd, Iran and Emeric Deutsch, Brooklyn, NY. (Vol. 51, No. 3, August 2013)**

Given a simple graph $G$ with vertex set $V$, a dominating set of $G$ is any subset $S$ of $V$ such that every vertex in $V\backslash S$ is adjacent to at least one vertex in $S$. Find the number of dominating sets of the path $P_n$ with $n$ vertices.

**Solution by Harris Kwong, Fredonia, NY.**

Let $d_n$ denote the number of dominating sets of $P_n$. Label the vertices of $P_n$ as $v_1, v_2, \ldots, v_n$ such that the vertices $v_i$ and $v_{i+1}$ are adjacent for $1 \le i \le n-1$. It is clear that $d_2 = 3$, because $P_2$ has three dominating sets: $\{v_1\}$, $\{v_2\}$, and $\{v_1, v_2\}$.

Consider $n \ge 3$. A dominating set $S$ may or may not contain $v_n$. If it does, then $S \setminus \{v_n\}$ is a dominating set of $P_{n-1}$. Conversely, any dominating set of $P_{n-1}$ can be expanded to a dominating set of $P_n$ by including $v_n$ in it. Hence, there are $d_{n-1}$ choices for $S$ in this case.

If $S$ does not contain $v_n$, it must contain $v_{n-1}$ so as to dominate $v_n$. Then $S \setminus \{v_{n-1}\}$ is a dominating set of $P_{n-2}$. Conversely, any dominating set of $P_{n-2}$ can be expanded to a dominating set of $P_n$ that also contains $v_{n-1}$ but not $v_n$ by adding $v_{n-1}$ to it. Thus, there are $d_{n-2}$ choices for $S$ in this case.

We have just proved that $d_n = d_{n-1} + d_{n-2}$ for $n \geq 3$. When $n = 3$, after we remove $v_2$ and $v_3$ in the second case, we are left with only one vertex $v_1$, which may or may not be contained in $S$. In this regard, we may define $d_1 = 2$. Along with $d_2 = 3$, we see that $d_n = F_{n+2}$.

**Also solved by the proposers.**

## An Application of the AM-GM Inequality

**H-741 Proposed by Charlie Cook, Sumter, South Carolina. (Vol. 51, No. 3, August 2013)**

If $n \geq 2$ and $m \geq 1$, then

$$m(L_n - F_n)(L_nF_n)^{(m-1)/2} \leq L_n^m - F_n^m,$$

where $L_n$ and $F_n$ are the Lucas and Fibonacci numbers, respectively.

**Solution by Ángel Plaza, Las Palmas, Spain.**

The proposed inequality is a particular case of the following more general inequality: If $0 < y \leq x$ and $m \geq 1$, then $m(x-y)(xy)^{(m-1)/2} \leq x^m - y^m$.

*Proof.* Last inequality may be written as

$$m(xy)^{(m-1)/2} \leq x^{m-1} + \cdots + x^{m-1-j}y^j + \cdots + y^{m-1}$$

$$(xy)^{(m-1)/2} \leq \frac{x^{m-1} + \cdots + x^{m-1-j}y^j + \cdots + y^{m-1}}{m}.$$

which follows immediately by the AM-GM inequality.

**Also solved by Kenneth B. Davenport, Dmitry Fleischman, Robinson Higuita, Harris Kwong, Hideyuki Ohtsuka, and the proposer.**

**Errata:** In problem **H-765**, the right-hand side of (iii) should be "$(L_nL_{n+1} - 2)^2$" instead of "$(L_nL_{n+1} - 1)^2$".

The published solution to **H-737** works for primes $p \geq 5$, but not for $p = 3$. Indeed, when $p$ is 3, $F_p$ is not necessarily prime to $L_{mp}$ and is not prime to $L_p$. Here is a fix from the same solver.

For $p = 3$, we have $\binom{3n-1}{3-1}_F = F_{3n-1}F_{3n-2} := a_n$, say. Then $a_n$ is a linear combination of $\alpha^{6n}$, $\beta^{6n}$ and $(-1)^n$, where $\alpha$ and $\beta$ are the zeros of $x^2 - x - 1$. Thus, $(a_n)$ is a linear recurrence with characteristic polynomial $(x^2 - L_6x + (\alpha\beta)^6)(x+1)$, which modulo 16 is $x^3 - x^2 - x + 1$. Therefore, we may prove that $a_n \equiv (-1)^{n-1} \pmod{F_3^2L_3 = 16}$ by induction as $a_{n+3} \equiv a_{n+2} + a_{n+1} - a_n \pmod{16}$.