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**PROBLEMS PROPOSED IN THIS ISSUE**

**H-751** Proposed by D. M. Bătinețu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.

Prove that
\[
\left( \frac{F_{m+1}^{2n} + F_{m+1}^{2n-1}F_m + \cdots + F_{m+1}^nF_m}{F_m} \right)^{p+1} + \left( \frac{F_{m+1}^nF_m^{n+1} + \cdots + F_{m+1}^{n+1}F_m^{2n-1}}{F_m} \right)^{p+1} \geq \frac{1}{2^p} \left( \frac{F_{m+2}^{2n+1}}{F_mF_{m+1}} \right)^{p+1}
\]
holds for any \( p \geq 0 \) and positive integers \( m \) and \( n \), and that the same inequality holds with all the \( F \)'s replaced by \( L \)'s.

**H-752** Proposed by D. M. Bătinețu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.

Prove that
\[
(1) \quad 5^m L_{2m+1} \sum_{p=0}^{2n+1} \binom{2n+1}{p} \sum_{k=0}^{p} \binom{p}{k} F_k = 5^n L_{2n+1} \sum_{p=0}^{2m+1} \binom{2m+1}{p} \sum_{k=0}^{p} \binom{p}{k} F_k,
\]
\[
(2) \quad 5^m F_{2m+1} \sum_{p=0}^{2n+1} \binom{2n+1}{p} \sum_{k=0}^{p} \binom{p}{k} L_k = 5^n L_{2n+1} \sum_{p=0}^{2m+1} \binom{2m+1}{p} \sum_{k=0}^{p} \binom{p}{k} L_k.
\]

**H-753** Proposed by H. Ohtsuka, Saitama, Japan.

For integers \( n \geq 1, \ m \geq 1, \ a \neq 0 \) and \( b \), prove that
\[
\sum_{k=1}^{n} F_{ak+b}^{4m} = \sum_{r=1}^{2m} \left( \frac{4m}{2m-r} \right) \frac{(-1)^{(an+b+1)r} F_{anr} L_{(an+a+2b)r}}{25^m F_{ar}} + \left( \frac{4m}{2m} \right) \frac{n}{25^m}.
\]
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H-754 Proposed by H. Ohtsuka, Saitama, Japan.

Let \( a, b \) and \( n \) be integers. The two sequences \( \{T_n\} \) and \( \{S_n\} \) satisfy
\[
T_{n+3} = T_{n+2} + T_{n+1} + T_n \quad \text{with arbitrary } T_0, T_1, T_2,
S_{n+3} = S_{n+2} + S_{n+1} + S_n \quad \text{with } S_0 = 3, S_1 = 1, S_2 = 3
\]
for all integers \( n \). Let \( R_n = S_n + 1 \). For \( n \geq 1 \), prove that
\[
(R_n^2 - R_{a-1}^2) \sum_{k=1}^{n} T_{ak+b}^2 = A_n - A_0,
\]
where
\[
A_n = 2T_{an+b}(R_aT_{an+a+b} + R_{-a}T_{an-a+b}) - (T_{an+a+b} - T_{an-a+b})^2 - (R_{-a}T_{an+b})^2.
\]

SOLUTIONS

Infinite Sums With Reciprocals of Squares of Fibonacci and Lucas Numbers

H-724 Proposed by H. Ohtsuka, Saitama, Japan.
(Vol. 50, No. 3, August 2012)

Determine
\[
\left( \sum_{k=1}^{\infty} \frac{1}{F_{4k}^2} - \sum_{k=1}^{\infty} \frac{1}{L_{4k}^2} + \sum_{k=1}^{\infty} \frac{1}{L_{2k}^2} \right) \left( \sum_{k=1}^{\infty} \frac{1}{F_{2k}^2} \right)^{-1}.
\]

Solution by the proposer.

The following identity is easily verified.
\[
5F_n^2 = L_n^2 - 4(-1)^n.
\]
If \( n \neq 0 \), dividing both sides of the identity by \( F_{2n}^2 \),
\[
5 \frac{L_n^2}{F_{2n}^2} = \frac{1}{F_{2n}^2} - 4(-1)^n.
\]

Using the above identity, we have
\[
\sum_{k=1}^{\infty} \frac{5}{F_{4k}^2} - \sum_{k=1}^{\infty} \frac{5}{L_{4k}^2} + \sum_{k=1}^{\infty} \frac{5}{L_{2k}^2} - \sum_{k=1}^{\infty} \frac{1}{F_{2k}^2}
= \sum_{k=1}^{\infty} \left( \frac{5}{F_{4k}^2} - \frac{5}{L_{4k}^2} \right) + \sum_{k=1}^{\infty} \left( \frac{5}{L_{2k}^2} - \frac{1}{F_{2k}^2} \right)
= \sum_{k=1}^{\infty} \left( \frac{5}{F_{4k}^2} - \frac{1}{F_{2k}^2} + \frac{4}{F_{2k+2}^2} \right) + \sum_{k=1}^{\infty} \left( -\frac{4}{F_{2k+2}^2} \right)
= 4 \sum_{k=1}^{\infty} \left( \frac{1}{F_{2k}^2} + \frac{1}{F_{2k+2}^2} \right) - 4 \sum_{k=1}^{\infty} \frac{1}{F_{2k+1}^2}
= 4 \sum_{k=2}^{\infty} \frac{1}{F_{2k}^2} - 4 \sum_{k=2}^{\infty} \frac{1}{F_{2k}^2} = 0.
\]
Thus, we have
\[ 5 \left( \sum_{k=1}^{\infty} \frac{1}{F_{2k}^4} - \sum_{k=1}^{\infty} \frac{1}{L_{4k}^2} + \sum_{k=1}^{\infty} \frac{1}{L_{2k}^2} \right) = \sum_{k=1}^{\infty} \frac{1}{F_{2k}^2}. \]
Therefore,
\[ \left( \sum_{k=1}^{\infty} \frac{1}{F_{2k}^4} - \sum_{k=1}^{\infty} \frac{1}{L_{4k}^2} + \sum_{k=1}^{\infty} \frac{1}{L_{2k}^2} \right) \left( \sum_{k=1}^{\infty} \frac{1}{F_{2k}^2} \right)^{-1} = \frac{1}{5}. \]

Also solved by Paul S. Bruckman and Dmitry Fleishman.

**Sums With Powers of \(-3, 4\) and Binomial Coefficients**

**H-725** Proposed by Paul S. Bruckman, Nanaimo, BC. (Vol. 50, No. 4, November 2012)

Prove the following identities valid for \(n = 0, 1, 2, \ldots\)

(a) \[ \sum_{k=0}^{[n/4]} \binom{n-3k}{k} (-3)^{3k} 4^{n-4k} = \frac{1}{6} \left( (3n+5)3^n - (-1)^n 3^{n/2} \frac{\sin((n-1)\theta)}{\sin \theta} \right), \]
where \(\sin \theta = \sqrt{2/3};\)

(b) \[ \sum_{k=0}^{[n/4]} \binom{n-k}{3k} (-3)^{3k} 4^{n-4k} = \frac{1}{18} \left( 9n + 7 + 3^{3n/2} (11 \cos(n\rho) + \sin(n\rho)/\sqrt{2}) \right), \]
where \(\sin \rho = \sqrt{2/27};\)

(c) \[ \sum_{k=0}^{[n/4]} \binom{n-2k}{2k} \left( \frac{(pq(p^2 - q^2))}{(p^2 + q^2)^2} \right)^{2k} = \frac{(p(p+q))^{n+1} - (q(q-p))^{n+1}}{2(p^2 + 2pq - q^2)(p^2 + q^2)^n} + \frac{(p(p-q))^{n+1} - (q(q+p))^{n+1}}{2(p^2 - 2pq - q^2)(p^2 + q^2)^n}, \]
where \(p > q > 0\) are integers.

**Solution by G. C. Greubel.**

(a) The process for the three series will be to consider the generating function of the given relations and compare the final results. For the first series in question consider
\[ S^1_n = \sum_{k=0}^{[n/4]} \binom{n-3k}{k} (-3)^{3k} 4^{n-4k}. \]
From this, we have
\[ \sum_{n=0}^{\infty} S^1_n t^n = \sum_{n,k=0}^{\infty} \binom{n+k}{k} (-3)^{3k} 4^{n+k+4k} = \sum_{k=0}^{\infty} (-3)^{3k} t^{4k} \cdot \sum_{n=0}^{\infty} \frac{(k+1)_n}{n!} (4t)^n = \sum_{k=0}^{\infty} \left( (-3)^3 t^4 \right)^k (1 - 4t)^{-k-1} = \frac{1}{1 - 4t} \sum_{k=0}^{\infty} \left( \frac{-27t^4}{(1 - 4t)} \right)^k = \frac{1}{1 - 4t + 27t^4}. \]
Alternatively, let \( \phi_n^1 \) be given by

\[
\phi_n^1 = \frac{1}{6} \left( (3n + 5)3^n - (-1)^n 3^{n/2}\frac{\sin(n-1)\theta}{\sin \theta} \right),
\]

for which

\[
\sum_{n=0}^{\infty} \phi_n^1 t^n = \frac{1}{6} \left[ \frac{5 - 6t}{(1 - 3t)^2} + \sum_{n=0}^{\infty} \frac{(-\sqrt{3})^n \sin(n - 1)\theta}{\sin \theta} \right].
\]

The summation on the right-hand side can be evaluated as follows.

\[
\sum_{n=0}^{\infty} (-\sqrt{3})^n \sin(n - 1)\theta = \frac{1}{2i} \left( \frac{e^{-i\theta}}{1 + \sqrt{3}e^{i\theta t}} - \frac{e^{i\theta}}{1 + \sqrt{3}e^{-i\theta t}} \right) = (-1) \sin \theta \left( \frac{1 + 2\sqrt{3} \cos \theta t}{1 + 2\sqrt{3} \cos \theta t + 3t^2} \right).
\]

From this, we have

\[
\sum_{n=0}^{\infty} \phi_n^1 t^n = \frac{1}{6} \left[ \frac{5 - 6t}{(1 - 3t)^2} + \frac{1 + 2\sqrt{3} \cos \theta t}{1 + 2\sqrt{3} \cos \theta t + 3t^2} \right].
\]

In order to reduce this expression use \( \sin \theta = \sqrt{2/3} \) or \( \cos \theta = \sqrt{1/3} \) to obtain

\[
\sum_{n=0}^{\infty} \phi_n^1 t^n = \frac{1}{6} \left[ \frac{5 - 6t}{1 - 6t + 9t^2} + \frac{1}{1 + 2t + 3t^2} \right] = \frac{1}{1 - 4t + 27t^2}.
\]

By comparing equations (1) and (2), we are led to the desired result

\[
\sum_{k=0}^{\lfloor n/4 \rfloor} \binom{n - 3k}{k} (-3)^k 4^{n-4k} = \frac{1}{6} \left( (3n + 5)3^n - (-1)^n 3^{n/2}\frac{\sin(n - 1)\theta}{\sin \theta} \right),
\]

where \( \sin \theta = \sqrt{2/3} \).

(b) Let \( S_n^2 \) be the series to be summed. As before, consider the generating function for this series.

\[
\sum_{n=0}^{\infty} S_n^2 t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/4]} \binom{n - k}{3k} (-3)^k 4^{n-4k} t^n = \sum_{n,k=0}^{\infty} \binom{n + 3k}{3k} (-3)^k 4^n t^{n+3k} = \sum_{k=0}^{\infty} (-3)^k t^{4k} \sum_{n=0}^{\infty} \frac{(3k + 1)n}{n!} (4t)^n = \frac{1}{1 - 4t} \sum_{k=0}^{\infty} \frac{(-27t^4)^k}{(1 - 4t)^3} = \frac{(1 - 4t)^2}{(1 - 4t)^3 + 27t^4}.
\]
Now that we have the desired generating function to compare to, let $\phi_n^2$ be the right-hand side of the desired result and proceed to find its generating function.

$$\sum_{n=0}^{\infty} \phi_n^2 t^n = \frac{1}{18} \sum_{n=0}^{\infty} \left(7 + 9n + \frac{3^{3n/2}}{\sqrt{2}} (11 \sqrt{2} \cos(n \rho) + \sin(n \rho)) \right) t^n$$

$$= \frac{1}{18} \left(\frac{7 + 2t}{(1 - t)^2} \right) + \frac{1}{18} \left(11 \gamma_1 + \frac{1}{\sqrt{2}} \gamma_2 \right),$$

where $\gamma_{1,2}$ are the series

$$\gamma_1 = \sum_{n=0}^{\infty} \cos(n \rho) (3^{3/2} t)^n, \quad \text{and} \quad \gamma_2 = \sum_{n=0}^{\infty} \sin(n \rho) (3^{3/2} t)^n.$$

Evaluating $\gamma_1$ yields

$$\gamma_1 = \frac{1}{2} \left[ \frac{1}{1 - 3^{3/2} e^{i \rho t}} + \frac{1}{1 - 3^{3/2} e^{-i \rho t}} \right] = \frac{1 - 3^{3/2} \cos \rho \cdot t}{1 - 2 \cdot 3^{3/2} \cos \rho \cdot t + 27 t^2}.$$ (5)

The result for $\gamma_2$ is similar and is given by

$$\gamma_2 = \frac{3^{3/2} \sin \rho \cdot t}{1 - 2 \cdot 3^{3/2} \cos \rho \cdot t + 27 t^2}.$$ (6)

By combining (4), (5) and (6), the resulting series takes the form

$$\sum_{n=0}^{\infty} \phi_n^2 t^n = \frac{1}{18} \left(\frac{7 + 2t}{(1 - t)^2} \right) + \frac{11 \sqrt{2} - 3^{3/2} (11 \sqrt{2} \cos \rho - \sin \rho) t}{\sqrt{2} (1 - 23^{3/2} \cos \rho t + 27 t^2)}.$$ (7)

Now using the provided value $\sin \rho = \sqrt{2/27}$, which also provides $\cos \rho = \sqrt{25/27}$, leads to the reduction

$$\sum_{n=0}^{\infty} \phi_n^2 t^n = \frac{1}{18} \left[ \frac{7 + 2t}{(1 - t)^2} + \frac{11 - 54t}{1 - 10t + 27 t^2} \right] = \frac{1}{18} \left[ \frac{18 - 144t + 268t^2}{1 - 12t + 48 t^2 - 64t^3 + 27 t^4} \right]$$

$$= \frac{(1 - 4t)^2}{(1 - 4t)^3 + 27 t^4}.$$ (8)

Since (3) and (8) have the same generating function, then we conclude that

$$\sum_{k=0}^{\lfloor n/4 \rfloor} \binom{n-k}{3k} (-3)^k 4^{n-4k} = \frac{1}{18} \left[ 7 + 9n + \frac{3^{3n/2}}{\sqrt{2}} (11 \sqrt{2} \cos(n \rho) + \sin(n \rho)) \right],$$ (9)

where $\sin \rho = \sqrt{2/27}$.

(c) Let

$$x = \frac{pq(p^2 - q^2)}{(p^2 + q^2)^2},$$ (10)
and let $S_n^3$ be the series in question, then
\[
\sum_{n=0}^{\infty} S_n^3 t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/4]} \left( \frac{n - 2k}{2k} \right) x^{2k} t^n = \sum_{n, k=0}^{\infty} \left( \frac{n + 2k}{2k} \right) (x^2 t^4)^k t^n
\]
\[
= \sum_{n, k=0}^{\infty} \left( \frac{2k + 1}{n!} \right) (x^2 t^4)^k t^n = \frac{1}{1 - t} \sum_{k=0}^{\infty} \left( \frac{x^2 t^4}{(1 - t)^2} \right)^k
\]
\[
= \frac{1 - t}{(1 - t)^2 - x^2 t^4}. \tag{11}
\]

Alternatively, let $\phi_n^3$ be the right-hand side of the third series in question and seek the generating function.
\[
\sum_{n=0}^{\infty} \phi_n^3 t^n = \frac{1}{2(p^2 + 2pq - q^2)} \Upsilon_3 + \frac{1}{2(p^2 - 2pq - q^2)} \Upsilon_4, \tag{12}
\]
where
\[
\Upsilon_3 = \sum_{n=0}^{\infty} \frac{(p(p + q))^{n+1} - (q(q - p))^{n+1}}{(p^2 + q^2)^n} t^n, \quad \text{and} \quad \Upsilon_4 = \sum_{n=0}^{\infty} \frac{(p(p - q))^{n+1} - (q(p + q))^{n+1}}{(p^2 + q^2)^n} t^n.
\]

Consider the evaluation of $\Upsilon_3$. This is done in the following way.
\[
\Upsilon_3 = \sum_{n=0}^{\infty} \frac{(p(p + q))^{n+1} - (q(q - p))^{n+1}}{(p^2 + q^2)^n} t^n = \frac{p(p + q)(p^2 + q^2)}{p^2 + q^2 - p(p + q)t} - \frac{q(q - p)(p^2 + q^2)}{p^2 + q^2 - q(q - p)t}
\]
\[
= \frac{(p^2 + 2pq - q^2)(p^2 + q^2)^2}{(p^2 + q^2)^2(1 - t) - pq(p^2 - q^2)t^2} = \frac{p^2 + 2pq - q^2}{1 - t - xt^2}, \tag{13}
\]
where $x$ is given by (10). The same process may be applied to evaluating $\Upsilon_4$ and we are led to the result
\[
\Upsilon_4 = \frac{p^2 - 2pq - q^2}{1 - t + xt^2}. \tag{14}
\]

From these expressions equation (12) is reduced to
\[
\sum_{n=0}^{\infty} \phi_n^3 t^n = \frac{1}{2} \left( \frac{1}{1 - t - xt^2} + \frac{1}{1 - t + xt^2} \right) = \frac{1 - t}{(1 - t)^2 - x^2 t^4}.
\]

This is the same generating function as that of equation (11). Thus leading to the statement
\[
\sum_{k=0}^{[n/4]} \left( \frac{n - 2k}{2k} \right) \left( \frac{pq(p^2 - q^2)}{(p^2 + q^2)^2} \right)^{2k} = \frac{(p(p + q))^{n+1} - (q(q - p))^{n+1}}{2(p^2 + 2pq - q^2)(p^2 + q^2)^n} + \frac{(p(p - q))^{n+1} - (q(q + p))^{n+1}}{2(p^2 - 2pq - q^2)(p^2 + q^2)^n}, \tag{15}
\]
where $p > q > 0$ are integers.

Also solved by Paul S. Bruckman and Kenneth B. Davenport.
Sums of Sums of Reciprocals of Fibonacci Numbers

H-726 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 50, No. 4, November 2012)

Prove that
\[
\sum_{k=1}^{\infty} \left( \frac{1}{F_{2k}} - \frac{1}{F_{4k}} + \frac{1}{F_{8k}} + \cdots + \frac{1}{F_{2^n k}} + \cdots \right) = \sum_{k=1}^{\infty} \frac{1}{F_{2k-1} F_{2k}}.
\]

Solution by the proposer.

First, we will prove the following lemma.

Lemma 1.
\[
\begin{align*}
(1) \quad & \frac{1}{F_n F_{n+1}} = \frac{1}{\alpha^n F_n} + \frac{1}{\alpha^{n+1} F_{n+1}}, \\
(2) \quad & \frac{1}{F_{2n}} = \frac{1}{\alpha^n F_n} - \frac{(-1)^n}{\alpha^{2n} F_{2n}}.
\end{align*}
\]

Proof of Lemma.

(1) We have
\[
\sqrt{5}(\alpha F_{n+1} + F_n) = \alpha(\alpha^{n+1} - \beta^{n+1}) + (\alpha^n - \beta^n)
\]
\[
= \alpha^{n+2} + \alpha^n = \alpha^n(\alpha^2 + 1) = \sqrt{5}\alpha^{n+1}.
\]

Thus,
\[
\alpha^{n+1} = \alpha F_{n+1} + F_n.
\]

Dividing both sides of this identity by \(\alpha^{n+1} F_{n+1}\), we get identity (1).

(2) We have
\[
\alpha^n L_n = \alpha^n(\alpha^n + \beta^n) = \alpha^{2n} + (-1)^n.
\]

Thus,
\[
\alpha^{2n} = \alpha^n L_n - (-1)^n.
\]

Dividing both sides of this identity by \(\alpha^{2n} F_{2n}\), we get identity (2). \(\Box\)

Using Lemma 1 (1), we have
\[
\sum_{k=1}^{\infty} \frac{1}{F_{2k-1} F_{2k}} = \sum_{k=1}^{\infty} \left( \frac{1}{\alpha F_{2k-1} F_{2k}} + \frac{1}{\alpha^{2k} F_{2k}} \right) = \sum_{k=1}^{\infty} \frac{1}{\alpha^k F_k}.
\]

Using Lemma 1 (2), we have
\[
\sum_{k=1}^{\infty} \frac{1}{F_{4k}} = \sum_{k=1}^{\infty} \frac{1}{\alpha^{2k} F_{2k}} - \sum_{k=1}^{\infty} \frac{1}{\alpha^{4k} F_{4k}} = \sum_{k=1}^{\infty} \frac{1}{\alpha^{4k-2} F_{4k-2}}.
\]
Using Lemma 1 (2), we have

\[
\sum_{n=3}^{\infty} \frac{1}{F_{2^k}^n} = \lim_{N \to \infty} \sum_{n=3}^{N} \left( \frac{1}{\alpha^{2^{n-1}k}F_{2^{n-1}k}} - \frac{1}{\alpha^{2^n k}F_{2^n k}} \right) \\
= \lim_{N \to \infty} \left( \frac{1}{\alpha^{4k}F_{4k}} - \frac{1}{\alpha^{2Nk}F_{2Nk}} \right) = \frac{1}{\alpha^{4k}F_{4k}}. \tag{3}
\]

We have

\[
\sum_{k=1}^{\infty} \frac{1}{F_{2k}} = \sum_{k=1}^{\infty} \frac{1}{\alpha^k F_k} - \sum_{k=1}^{\infty} \frac{(-1)^k}{\alpha^{2k} F_{2k}} \quad \text{(by Lemma 1 (2))}
\]

\[
= \sum_{k=1}^{\infty} \frac{1}{F_{2k-1}F_{2k}} - \sum_{k=1}^{\infty} \frac{1}{\alpha^{4k}F_{4k}} + \sum_{k=1}^{\infty} \frac{1}{\alpha^{4k-2}F_{4k-2}} \quad \text{(by (1))}
\]

\[
= \sum_{k=1}^{\infty} \frac{1}{F_{2k-1}F_{2k}} - \sum_{k=1}^{\infty} \sum_{n=3}^{\infty} \frac{1}{F_{2^n k}} + \sum_{k=1}^{\infty} \frac{1}{F_{4k}} \quad \text{(by (2) and (3)).}
\]

Thus, we obtain,

\[
\sum_{k=1}^{\infty} \frac{1}{F_{2k}} - \sum_{k=1}^{\infty} \frac{1}{F_{4k}} + \sum_{k=1}^{\infty} \sum_{n=3}^{\infty} \frac{1}{F_{2^n k}} = \sum_{k=1}^{\infty} \frac{1}{F_{2k-1}F_{2k}}.
\]

Also solved by Paul S. Bruckman.