ADVANCED PROBLEMS AND SOLUTIONS

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PROBLEMS PROPOSED IN THIS ISSUE

H-751 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.

Prove that

$$\left(\frac{F_{m+1}^{2n}+\binom{2n+1}{1}F_{m+1}^{2n-1}F_m+\cdots+\binom{2n+1}{n}F_{m+1}^nF_m^n}{F_m}\right)^{p+1}$$
$$+\left(\frac{\binom{2n+1}{n+1}F_{m+1}^nF_m^n+\cdots+\binom{2n+1}{2n}F_{m+1}F_m^{2n-1}}{F_{m+1}}\right)^{p+1}\geq\frac{1}{2^p}\left(\frac{F_{m+2}^{2n+1}}{F_mF_{m+1}}\right)^{p+1}$$

holds for any $p\geq 0$ and positive integers m and n, and that the same inequality holds with all the F's replaced by L's.

H-752 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.

Prove that

$$(1)\quad 5^mL_{2m+1}\sum_{p=0}^{2n+1}\binom{2n+1}{p}\sum_{k=0}^{p}\binom{p}{k}F_k=5^nL_{2n+1}\sum_{p=0}^{2m+1}\binom{2m+1}{p}\sum_{k=0}^{p}\binom{p}{k}F_k,$$

$$(2)\quad 5^mF_{2m+1}\sum_{p=0}^{2n+1}\binom{2n+1}{p}\sum_{k=0}^{p}\binom{p}{k}L_k=5^nF_{2n+1}\sum_{p=0}^{2m+1}\binom{2m+1}{p}\sum_{k=0}^{p}\binom{p}{k}L_k.$$

H-753 Proposed by H. Ohtsuka, Saitama, Japan.

For integers $n\geq 1,\ m\geq 1,\ a\neq 0$ and b, prove that

$$\sum_{k=1}^{n}F_{ak+b}^{4m}=\sum_{r=1}^{2m}\binom{4m}{2m-r}\frac{(-1)^{(an+b+1)r}F_{anr}L_{(an+a+2b)r}}{25^mF_{ar}}+\binom{4m}{2m}\frac{n}{25^m}.$$

H-754 Proposed by H. Ohtsuka, Saitama, Japan.

Let $a,\ b$ and n be integers. The two sequences $\{T_n\}$ and $\{S_n\}$ satisfy

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n \quad \text{with arbitrary } T_0,\ T_1,\ T_2,$$
$$S_{n+3} = S_{n+2} + S_{n+1} + S_n \quad \text{with } S_0 = 3,\ S_1 = 1,\ S_2 = 3$$

for all integers n. Let $R_n = S_n + 1$. For $n \geq 1$, prove that

$$(R_a^2 - R_{-a}^2) \sum_{k=1}^{n} T_{ak+b}^2 = A_n - A_0,$$

where

$$A_n = 2T_{an+b}(R_a T_{an+a+b} + R_{-a} T_{an-a+b}) - (T_{an+a+b} - T_{an-a+b})^2 - (R_{-a} T_{an+b})^2.$$

SOLUTIONS

Infinite Sums With Reciprocals of Squares of Fibonacci and Lucas Numbers

H-724 Proposed by H. Ohtsuka, Saitama, Japan.
(Vol. 50, No. 3, August 2012)

Determine

$$\left(\sum_{k=1}^{\infty} \frac{1}{F_{4^k}^2} - \sum_{k=1}^{\infty} \frac{1}{L_{4^k}^2} + \sum_{k=1}^{\infty} \frac{1}{L_{2^k}^2} \right) \left(\sum_{k=1}^{\infty} \frac{1}{F_{2^k}^2} \right)^{-1}.$$

Solution by the proposer.

The following identity is easily verified.

$$5F_n^2 = L_n^2 - 4(-1)^n.$$

If $n \neq 0$, dividing both sides of the identity by F_{2n}^2,

$$\frac{5}{L_n^2} = \frac{1}{F_n^2} - \frac{4(-1)^n}{F_{2n}^2}.$$

Using the above identity, we have

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{5}{F_{4^k}^2} - \sum_{k=1}^{\infty} \frac{5}{L_{4^k}^2} + \sum_{k=1}^{\infty} \frac{5}{L_{2^k}^2} - \sum_{k=1}^{\infty} \frac{1}{F_{2^k}^2} \\
&= \sum_{k=1}^{\infty} \left(\frac{5}{F_{4^k}^2} - \frac{5}{L_{4^k}^2} \right) + \sum_{k=1}^{\infty} \left(\frac{5}{L_{2^k}^2} - \frac{1}{F_{2^k}^2} \right) \\
&= \sum_{k=1}^{\infty} \left(\frac{5}{F_{4^k}^2} - \frac{1}{F_{4^k}^2} + \frac{4}{F_{2\cdot 4^k}^2} \right) + \sum_{k=1}^{\infty} \left(-\frac{4}{F_{2\cdot 2^k}^2} \right) \\
&= 4\sum_{k=1}^{\infty} \left(\frac{1}{F_{2^{2k}}^2} + \frac{1}{F_{2^{2k+1}}^2} \right) - 4\sum_{k=1}^{\infty} \frac{1}{F_{2^{k+1}}^2} \\
&= 4\sum_{k=2}^{\infty} \frac{1}{F_{2^k}^2} - 4\sum_{k=2}^{\infty} \frac{1}{F_{2^k}^2} = 0.
\end{aligned}$$

Thus, we have

$$5\left(\sum_{k=1}^{\infty}\frac{1}{F_{4^k}^2}-\sum_{k=1}^{\infty}\frac{1}{L_{4^k}^2}+\sum_{k=1}^{\infty}\frac{1}{L_{2^k}^2}\right)=\sum_{k=1}^{\infty}\frac{1}{F_{2^k}^2}.$$

Therefore,

$$\left(\sum_{k=1}^{\infty}\frac{1}{F_{4^k}^2}-\sum_{k=1}^{\infty}\frac{1}{L_{4^k}^2}+\sum_{k=1}^{\infty}\frac{1}{L_{2^k}^2}\right)\left(\sum_{k=1}^{\infty}\frac{1}{F_{2^k}^2}\right)^{-1}=\frac{1}{5}.$$

Also solved by Paul S. Bruckman and Dmitry Fleishman.

Sums With Powers of -3, 4 and Binomial Coefficients

H-725 Proposed by Paul S. Bruckman, Nanaimo, BC.
(Vol. 50, No. 4, November 2012)

Prove the following identities valid for $n = 0, 1, 2, \ldots$

(a) $$\sum_{k=0}^{\lfloor n/4\rfloor}\binom{n-3k}{k}(-3)^{3k}4^{n-4k}=\frac{1}{6}\left((3n+5)3^n-(-1)^n 3^{n/2}\frac{\sin((n-1)\theta)}{\sin\theta}\right),$$
where $\sin\theta=\sqrt{2/3}$;

(b) $$\sum_{k=0}^{\lfloor n/4\rfloor}\binom{n-k}{3k}(-3)^{3k}4^{n-4k}=\frac{1}{18}\left(9n+7+3^{3n/2}(11\cos(n\rho)+\sin(n\rho)/\sqrt{2})\right),$$
where $\sin\rho=\sqrt{2/27}$;

(c) $$\sum_{k=0}^{\lfloor n/4\rfloor}\binom{n-2k}{2k}\left(\frac{(pq(p^2-q^2))}{(p^2+q^2)^2}\right)^{2k}=\frac{(p(p+q))^{n+1}-(q(q-p))^{n+1}}{2(p^2+2pq-q^2)(p^2+q^2)^n}$$
$$+\ \frac{(p(p-q))^{n+1}-(q(q+p))^{n+1}}{2(p^2-2pq-q^2)(p^2+q^2)^n},$$ where $p>q>0$ are integers.

Solution by G. C. Greubel.

(a) The process for the three series will be to consider the generating function of the given relations and compare the final results. For the first series in question consider

$$S_n^1=\sum_{k=0}^{\lfloor n/4\rfloor}\binom{n-3k}{k}(-3)^{3k}4^{n-4k}.$$

From this, we have

$$\begin{aligned}\sum_{n=0}^{\infty}S_n^1t^n&=\sum_{n,k=0}^{\infty}\binom{n+k}{k}(-3)^{3k}4^nt^{n+4k}=\sum_{k=0}^{\infty}(-3)^{3k}t^{4k}\cdot\sum_{n=0}^{\infty}\frac{(k+1)_n}{n!}(4t)^n\\&=\sum_{k=0}^{\infty}\left((-3)^3t^4\right)^k(1-4t)^{-k-1}=\frac{1}{1-4t}\sum_{k=0}^{\infty}\left(\frac{-27t^4}{(1-4t)}\right)^k\\&=\frac{1}{1-4t+27t^4}.\end{aligned}\tag{1}$$

Alternatively, let ϕ_n^1 be given by

$$\phi_n^1 = \frac{1}{6}\left((3n+5)3^n - (-1)^n 3^{n/2}\frac{\sin(n-1)\theta}{\sin\theta}\right),$$

for which

$$\sum_{n=0}^{\infty} \phi_n^1 t^n = \frac{1}{6}\left[\frac{5-6t}{(1-3t)^2} - \sum_{n=0}^{\infty}\frac{(-\sqrt{3}t)^n \sin(n-1)\theta}{\sin\theta}\right].$$

The summation on the right-hand side can be evaluated as follows.

$$\sum_{n=0}^{\infty}(-\sqrt{3}t)^n \sin(n-1)\theta = \frac{1}{2i}\sum_{n=0}^{\infty}\left(e^{i(n-1)\theta} - e^{-i(n-1)\theta}\right)(-\sqrt{3}t)^n$$

$$= \frac{1}{2i}\left(\frac{e^{-i\theta}}{1+\sqrt{3}e^{i\theta}t} - \frac{e^{i\theta}}{1+\sqrt{3}e^{-i\theta}t}\right) = (-1)\sin\theta\left(\frac{1+2\sqrt{3}\cos\theta t}{1+2\sqrt{3}\cos\theta t+3t^2}\right).$$

From this, we have

$$\sum_{n=0}^{\infty} \phi_n^1 t^n = \frac{1}{6}\left[\frac{5-6t}{(1-3t)^2} + \frac{1+2\sqrt{3}\cos\theta t}{1+2\sqrt{3}\cos\theta t+3t^2}\right].$$

In order to reduce this expression use $\sin\theta = \sqrt{2/3}$ or $\cos\theta = \sqrt{1/3}$ to obtain

$$\sum_{n=0}^{\infty} \phi_n^1 t^n = \frac{1}{6}\left[\frac{5-6t}{1-6t+9t^2} + \frac{1+2t}{1+2t+3t^2}\right] = \frac{1}{1-4t+27t^2}. \tag{2}$$

By comparing equations (1) and (2), we are led to the desired result

$$\sum_{k=0}^{\lfloor n/4\rfloor}\binom{n-3k}{k}(-3)^{3k}4^{n-4k} = \frac{1}{6}\left((3n+5)3^n - (-1)^n 3^{n/2}\frac{\sin(n-1)\theta}{\sin\theta}\right),$$

where $\sin\theta = \sqrt{2/3}$.

(b) Let S_n^2 be the series to be summed. As before, consider the generating function for this series.

$$\sum_{n=0}^{\infty} S_n^2 t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/4]}\binom{n-k}{3k}(-3)^{3k}4^{n-4k}t^n = \sum_{n,k=0}^{\infty}\binom{n+3k}{3k}(-3)^{3k}4^n t^{n+3k}$$

$$= \sum_{k=0}^{\infty}(-3)^{3k}t^{4k}\cdot\sum_{n=0}^{\infty}\frac{(3k+1)_n}{n!}(4t)^n = \frac{1}{1-4t}\sum_{k=0}^{\infty}\left(\frac{-27t^4}{(1-4t)^3}\right)^k$$

$$= \frac{(1-4t)^2}{(1-4t)^3+27t^4}. \tag{3}$$

Now that we have the desired generating function to compare to, let ϕ_n^2 be the right-hand side of the desired result and proceed to find its generating function.

$$\sum_{n=0}^{\infty}\phi_n^2 t^n = \frac{1}{18}\sum_{n=0}^{\infty}\left(7+9n+\frac{3^{3n/2}}{\sqrt{2}}(11\sqrt{2}\cos(n\rho)+\sin(n\rho))\right)t^n$$
$$= \frac{1}{18}\left(\frac{7+2t}{(1-t)^2}\right)+\frac{1}{18}\left(11\Upsilon_1+\frac{1}{\sqrt{2}}\Upsilon_2\right), \tag{4}$$

where $\Upsilon_{1,2}$ are the series

$$\Upsilon_1 = \sum_{n=0}^{\infty}\cos(n\rho)(3^{3/2}t)^n, \quad \text{and} \quad \Upsilon_2 = \sum_{n=0}^{\infty}\sin(n\rho)(3^{3/2}t)^n.$$

Evaluating Υ_1 yields

$$\Upsilon_1 = \frac{1}{2}\left[\frac{1}{1-3^{3/2}e^{i\rho}t}+\frac{1}{1-3^{3/2}e^{-i\rho}t}\right] = \frac{1-3^{3/2}\cos\rho\cdot t}{1-2\cdot 3^{3/2}\cos\rho\cdot t+27t^2}. \tag{5}$$

The result for Υ_2 is similar and is given by

$$\Upsilon_2 = \frac{3^{3/2}\sin\rho\cdot t}{1-2\cdot 3^{3/2}\cos\rho\cdot t+27t^2}. \tag{6}$$

By combining (4), (5) and (6), the resulting series takes the form

$$\sum_{n=0}^{\infty}\phi_n^2 t^n = \frac{1}{18}\left(\frac{7+2t}{(1-t)^2}\right)+\frac{11\sqrt{2}-3^{3/2}(11\sqrt{2}\cos\rho-\sin\rho)t}{\sqrt{2}(1-23^{3/2}\cos\rho t+27t^2)}. \tag{7}$$

Now using the provided value $\sin\rho = \sqrt{2/27}$, which also provides $\cos\rho = \sqrt{25/27}$, leads to the reduction

$$\sum_{n=0}^{\infty}\phi_n^2 t^n = \frac{1}{18}\left[\frac{7+2t}{(1-t)^2}+\frac{11-54t}{1-10t+27t^3}\right] = \frac{1}{18}\left[\frac{18-144t+268t^2}{1-12t+48t^2-64t^3+27t^4}\right]$$
$$= \frac{(1-4t)^2}{(1-4t)^3+27t^4}. \tag{8}$$

Since (3) and (8) have the same generating function, then we conclude that

$$\sum_{k=0}^{\lfloor n/4\rfloor}\binom{n-k}{3k}(-3)^{3k}4^{n-4k} = \frac{1}{18}\left(7+9n+\frac{3^{3n/2}}{\sqrt{2}}(11\sqrt{2}\cos(n\rho)+\sin(n\rho))\right), \tag{9}$$

where $\sin\rho = \sqrt{2/27}$.

(c) Let

$$x = \frac{pq(p^2-q^2)}{(p^2+q^2)^2}, \tag{10}$$

and let S_n^3 be the series in question, then

$$\begin{aligned}\sum_{n=0}^{\infty} S_n^3 t^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{[n/4]} \binom{n-2k}{2k} x^{2k} t^n = \sum_{n,k=0}^{\infty} \binom{n+2k}{2k} (x^2t^4)^k t^n \\ &= \sum_{n,k=0}^{\infty} \frac{(2k+1)_n}{n!} (x^2t^4)^k t^n = \frac{1}{1-t} \sum_{k=0}^{\infty} \left(\frac{x^2t^4}{(1-t)^2}\right)^k \\ &= \frac{1-t}{(1-t)^2 - x^2t^4}. \qquad (11)\end{aligned}$$

Alternatively, let ϕ_n^3 be the right-hand side of the third series in question and seek the generating function.

$$\sum_{n=0}^{\infty} \phi_n^3 t^n = \frac{1}{2(p^2+2pq-q^2)} \Upsilon_3 + \frac{1}{2(p^2-2pq-q^2)} \Upsilon_4, \qquad (12)$$

where

$$\Upsilon_3 = \sum_{n=0}^{\infty} \frac{(p(p+q))^{n+1} - (q(q-p))^{n+1}}{(p^2+q^2)^n} t^n, \quad \text{and} \quad \Upsilon_4 = \sum_{n=0}^{\infty} \frac{(p(p-q))^{n+1} - (q(p+q))^{n+1}}{(p^2+q^2)^n} t^n.$$

Consider the evaluation of Υ_3. This is done in the following way.

$$\begin{aligned}\Upsilon_3 &= \sum_{n=0}^{\infty} \frac{(p(p+q))^{n+1} - (q(q-p))^{n+1}}{(p^2+q^2)^n} \cdot t^n = \frac{p(p+q)(p^2+q^2)}{p^2+q^2-p(p+q)t} - \frac{q(q-p)(p^2+q^2)}{p^2+q^2-q(q-p)t} \\ &= \frac{(p^2+2pq-q^2)(p^2+q^2)^2}{(p^2+q^2)^2(1-t) - pq(p^2-q^2)t^2} = \frac{p^2+2pq-q^2}{1-t-xt^2}, \qquad (13)\end{aligned}$$

where x is given by (10). The same process may be applied to evaluating Υ_4 and we are led to the result

$$\Upsilon_4 = \frac{p^2-2pq-q^2}{1-t+xt^2}. \qquad (14)$$

From these expressions equation (12) is reduced to

$$\sum_{n=0}^{\infty} \phi_n^3 t^n = \frac{1}{2}\left(\frac{1}{1-t-xt^2} + \frac{1}{1-t+xt^2}\right) = \frac{1-t}{(1-t)^2 - x^2t^4}.$$

This is the same generating function as that of equation (11). Thus leading to the statement

$$\begin{aligned}\sum_{k=0}^{\lfloor n/4 \rfloor} \binom{n-2k}{2k} \left(\frac{pq(p^2-q^2)}{(p^2+q^2)^2}\right)^{2k} &= \frac{(p(p+q))^{n+1} - (q(q-p))^{n+1}}{2(p^2+2pq-q^2)(p^2+q^2)^n} \\ &\quad + \frac{(p(p-q))^{n+1} - (q(q+p))^{n+1}}{2(p^2-2pq-q^2)(p^2+q^2)^n}, \qquad (15)\end{aligned}$$

where $p > q > 0$ are integers.

Also solved by Paul S. Bruckman and Kenneth B. Davenport.

Sums of Sums of Reciprocals of Fibonacci Numbers

H-726 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 50, No. 4, November 2012)

Prove that

$$\sum_{k=1}^{\infty}\left(\frac{1}{F_{2k}}-\frac{1}{F_{4k}}+\frac{1}{F_{8k}}+\frac{1}{F_{16k}}+\cdots+\frac{1}{F_{2^n k}}+\cdots\right)=\sum_{k=1}^{\infty}\frac{1}{F_{2k-1}F_{2k}}.$$

Solution by the proposer.

First, we will prove the following lemma.

Lemma 1.

(1) $\dfrac{1}{F_nF_{n+1}}=\dfrac{1}{\alpha^nF_n}+\dfrac{1}{\alpha^{n+1}F_{n+1}}$;

(2) $\dfrac{1}{F_{2n}}=\dfrac{1}{\alpha^nF_n}-\dfrac{(-1)^n}{\alpha^{2n}F_{2n}}$.

Proof of Lemma.

(1) We have

$$\sqrt{5}(\alpha F_{n+1}+F_n)=\alpha(\alpha^{n+1}-\beta^{n+1})+(\alpha^n-\beta^n)$$
$$=\alpha^{n+2}+\alpha^n=\alpha^n(\alpha^2+1)=\sqrt{5}\alpha^{n+1}.$$

Thus,

$$\alpha^{n+1}=\alpha F_{n+1}+F_n.$$

Dividing both sides of this identity by $\alpha^{n+1}F_nF_{n+1}$, we get identity (1).

(2) We have

$$\alpha^nL_n=\alpha^n(\alpha^n+\beta^n)=\alpha^{2n}+(-1)^n.$$

Thus,

$$\alpha^{2n}=\alpha^nL_n-(-1)^n.$$

Dividing both sides of this identity by $\alpha^{2n}F_{2n}$, we get identity (2). □

Using Lemma 1 (1), we have

$$\sum_{k=1}^{\infty}\frac{1}{F_{2k-1}F_{2k}}=\sum_{k=1}^{\infty}\left(\frac{1}{\alpha^{2k-1}F_{2k-1}}+\frac{1}{\alpha^{2k}F_{2k}}\right)=\sum_{k=1}^{\infty}\frac{1}{\alpha^kF_k}. \tag{1}$$

Using Lemma 1 (2), we have

$$\sum_{k=1}^{\infty}\frac{1}{F_{4k}}=\sum_{k=1}^{\infty}\frac{1}{\alpha^{2k}F_{2k}}-\sum_{k=1}^{\infty}\frac{1}{\alpha^{4k}F_{4k}}=\sum_{k=1}^{\infty}\frac{1}{\alpha^{4k-2}F_{4k-2}}. \tag{2}$$

Using Lemma 1 (2), we have

$$\sum_{n=3}^{\infty}\frac{1}{F_{2^n k}} = \lim_{N\to\infty}\sum_{n=3}^{N}\left(\frac{1}{\alpha^{2^{n-1}k}F_{2^{n-1}k}} - \frac{1}{\alpha^{2^n k}F_{2^n k}}\right)$$
$$= \lim_{N\to\infty}\left(\frac{1}{\alpha^{4k}F_{4k}} - \frac{1}{\alpha^{2^N k}F_{2^N k}}\right) = \frac{1}{\alpha^{4k}F_{4k}}. \qquad (3)$$

We have

$$\sum_{k=1}^{\infty}\frac{1}{F_{2k}} = \sum_{k=1}^{\infty}\frac{1}{\alpha^k F_k} - \sum_{k=1}^{\infty}\frac{(-1)^k}{\alpha^{2k}F_{2k}} \qquad \text{(by Lemma 1 (2))}$$
$$= \sum_{k=1}^{\infty}\frac{1}{F_{2k-1}F_{2k}} - \sum_{k=1}^{\infty}\frac{1}{\alpha^{4k}F_{4k}} + \sum_{k=1}^{\infty}\frac{1}{\alpha^{4k-2}F_{4k-2}} \qquad \text{(by (1))}$$
$$= \sum_{k=1}^{\infty}\frac{1}{F_{2k-1}F_{2k}} - \sum_{k=1}^{\infty}\sum_{n=3}^{\infty}\frac{1}{F_{2^n k}} + \sum_{k=1}^{\infty}\frac{1}{F_{4k}}. \qquad \text{(by (2) and (3))}.$$

Thus, we obtain,

$$\sum_{k=1}^{\infty}\frac{1}{F_{2k}} - \sum_{k=1}^{\infty}\frac{1}{F_{4k}} + \sum_{k=1}^{\infty}\sum_{n=3}^{\infty}\frac{1}{F_{2^n k}} = \sum_{k=1}^{\infty}\frac{1}{F_{2k-1}F_{2k}}.$$

Also solved by Paul S. Bruckman.