ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, IMATE, UNAM, AP. POSTAL 61-3 (XANGARI), CP 58 089, MORELIA, MICHOACAN, MEXICO, or by e-mail at fluca@matmor.unam.mx as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-661 Proposed by J. López González, Madrid, Spain and F. Luca, Mexico

Let $\phi(n)$ and $\sigma(n)$ be the Euler function of n and the sum of divisors function of n, respectively.

(i) If n is odd perfect show that $0.4601 < \phi(n)/n < 0.5$.

(ii) Show that n is odd perfect if and only if $n\sigma(2n) = \sigma(n)(n + \sigma(n))$.

H-662 Proposed by Rigoberto Flórez, Sumter, SC

Let $T_n = n(n+1)/2$ be the nth triangular number.

(i) If $n \geq 1$, show that

$$n! = 2^{\lfloor n/2 \rfloor} \prod_{i=1}^{\lfloor n/2 \rfloor - 1} (T_{\lfloor (n+1)/2 \rfloor} - T_i).$$

(ii) Suppose that T is equal to $\prod_{i=0}^{k-1}(T_k - T_i)$ or $\prod_{i=0}^{k-1}(T_{k+1} - T_i)$. Let p be the first prime number greater than T. Is $p - T$ always equal to one or to a prime number?

H-663 Proposed by Charles K. Cook, Sumter, SC

If $n \geq 3$, evaluate $\prod_{j=1}^{F_n - 1} \sin(j\pi/F_n)$.

H-664 Proposed by A. Cusumano, Great Neck, NY

If a is a positive integer, prove that

$$\lim_{n\to\infty}(F_n^{1/a} + F_{n+a}^{1/a} - F_{n+2a}^{1/a}) = 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{F_n^{1/a} + F_{n+a}^{1/a}}{F_{n+2a}^{1/a}} = 1.$$

SOLUTIONS

A Fibonacci System

H-644 Proposed by José Luis Díaz-Barrero, Barcelona, Spain (Vol. 44, No. 3, August 2006)

Let n be a positive integer. Solve the following system of equations

$$\begin{pmatrix} 1+\frac{1}{F_1} & 1 & \dots & 1 \\ 1 & 1+\frac{1}{F_2} & \dots & 1 \\ \vdots & \vdots & \dots & \vdots \\ 1 & 1 & \dots & 1+\frac{1}{F_n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{pmatrix}.$$

Solution by H.-J. Seiffert, Berlin, Germany

The system of equations may be rewritten as

$$\frac{x_k}{F_k} + \sum_{j=1}^{n} x_j = F_k, \quad k = 1, 2, \dots, n.$$

Letting $S_n = \sum_{j=1}^{n} x_j$ and multiplying the kth equation by F_k yields

$$x_k + F_k S_n = F_k^2, \quad k = 1, 2, \dots, n. \tag{1}$$

Summing over $k = 1, 2, \dots, n$ and using the known summation formulas

$$\sum_{k=1}^{n} F_k = F_{n+2} - 1 \ \text{ and } \ \sum_{k=1}^{n} F_k^2 = F_n F_{n+1},$$

it follows easily that $S_n = F_n F_{n+1}/F_{n+2}$. Now, (1) gives the solution

$$x_k = F_k^2 - \frac{F_k F_n F_{n+1}}{F_{n+2}}, \quad k = 1, 2, \dots, n.$$

Also solved by Paul S. Bruckman, Rigoberto Flórez and Maitland Rose (jointly), Rebecca A. Hillman and Charles K. Cook (jointly), Harris Kwong and the proposer.

Mersenne and Fermat Factors of Lucas Numbers

H-645 Proposed by John H. Jaroma, Loyola College in Maryland, Baltimore, MD (Vol. 44, No. 3, August 2006)

(1) Show that every Mersenne prime is a factor of infinitely many L_n.

(2) Show that no Fermat prime is a factor of any L_n.

Solution by the proposer

(1) Let $2^p - 1$ be a Mersenne prime. Now, $\{F_n\}$ and $\{L_n\}$ are produced by the Lucas sequence $X_{n+2} = PX_{n+1} - QX_n$, $n \in \{0, 1, \dots\}$, where $P = 1$ and $Q = -1$. Let $D = P^2 - 4Q = 5$ be the discriminant of the characteristic equation. Since $p \geq 2$, we have $\left(\frac{Q}{2^p - 1}\right) = -1$. We recall that the *rank of apparition of a prime* p is the index of the first term in the underlying sequence that contains p as a divisor and that p divides infinitely many Lucas numbers if and only if its rank of apparition is even.

Case I. Suppose that $p \equiv 3 \pmod 4$. Thus,

$$\left(\frac{5}{2^p - 1}\right) = \left(\frac{2^p - 1}{5}\right) = \left(\frac{8 - 1}{5}\right) = \left(\frac{2}{5}\right) = -1,$$

where we used the fact that $2^4 \equiv 1 \pmod 5$. It now follows that $2^p - 1 \mid F_{2^p}$, so the rank of apparition of $2^p - 1$ is a divisor of 2^p. In particular, it is even, therefore $2^p - 1$ divides infinitely many Lucas numbers.

Case II. Suppose that $p \equiv 1 \pmod 4$. Then

$$\left(\frac{5}{2^p - 1}\right) = \left(\frac{2^p - 1}{5}\right) = \left(\frac{2 - 1}{5}\right) = 1,$$

therefore, $2^p - 1 \mid F_{2(2^{p-1}-1)}$. However, as $\left(\frac{Q}{2^p - 1}\right) = -1$, we get that $2^p - 1$ does not divide $F_{2^{p-1}-1}$. Hence, the rank of apparition of $2^p - 1$ in $\{F_n\}$ is even, therefore $2^p - 1$ divides infinitely many Lucas numbers.

(2) We may assume that the Fermat number in question exceeds 5 since it is easy to check that no Lucas number is a multiple of 5. It is well-known that 5 is not a quadratic residue modulo the Fermat number $2^{2^n} + 1$ for all $n \geq 2$. Indeed, this is used in Pepin's test for the primality of the Fermat number. Hence, if $2^{2^n} + 1$ is prime, then it divides $F_{2^{2^n}+2}$. Since

$\left(\frac{Q}{2^{2^n}+1}\right) = 1$, it follows that $2^{2^n}+1 \mid F_{2^{2^n-1}+1}$. Thus, the ranks of apparition of Fermat primes exceeding 5 in $\{F_n\}$ are odd, therefore, such numbers do not divide Lucas numbers.

Also solved by Paul S. Bruckman.

Variants of Wiefrich Numbers

H-646 Proposed by John H. Jaroma, Loyola College in Maryland, Baltimore, MD (Vol. 44, No. 3, August 2006)

A *Wiefrich prime* is any prime p that satisfies $2^{p-1} \equiv 1 \pmod{p^2}$. Presently, 1093 and 3511 are the only known such primes. Similarly defined is a *Wall-Sun-Sun prime*, which is any prime p such that $F_{p-(5/p)} \equiv 0 \pmod{p^2}$, where $(5/p)$ is the Legendre symbol. There are no known Wall-Sun-Sun primes. More generally, in 1993 P. Montgomery added 23 new solutions to $a^{p-1} \equiv 1 \pmod{p^2}$. This brought to 219 the number of observed solutions to $2 \le a \le 99$ and $3 \le p < 2^{32}$.

Let p be a prime and $n \ge 1$. Prove that there exist infinitely many integers $a > 1$ such that $a^{p-1} \equiv 1 \pmod{p^n}$.

Solution by Paul Young, Charleston, SC

For any prime p the integer $a = k \cdot p^n + 1$ is such an integer for any $k > 0$, since $a \equiv 1 \pmod{p^n}$ implies $a^{p-1} \equiv 1 \pmod{p^n}$. This shows that there are in fact infinitely many *primes* a satisfying the given condition (by Dirichlet's Theorem). For other solutions, observe that since the multiplicative group $(\boldsymbol{Z}/p^n\boldsymbol{Z})^\times$ of units of $\boldsymbol{Z}/p^n\boldsymbol{Z}$ has order $\phi(p^n) = p^{n-1}(p-1)$, we get that $(b^{p^{n-1}})^{p-1} \equiv 1 \pmod{p^n}$ for any integer b which is relatively prime to p. Thus, $a = b^{p^{n-1}}$ is such an integer for any integer $b > 1$ which is relatively prime to p, as is any positive integer of the form $a = b^{p^{n-1}} + k \cdot p^n$, where $(b,p) = 1$ and $k \in \boldsymbol{Z}$. There are therefore infinitely many such primes a in every nontrivial residue class modulo p.

Also solved by Paul S. Bruckman and the proposer.

Summing Members of a Binary Recurrence

H-647 Proposed by N. Gauthier, Kingston, ON (Vol. 44, No. 4, November 2006)

For a positive integer n and a non-zero number k, consider the following recurrence for generalized Fibonacci numbers

$$f_{r+2} = kf_{r+1} + f_r; \qquad r \ge 0; \qquad f_0 = 0,\ f_1 = 1.$$

a) Prove the following two identities

$$\sum_{r=0}^{n-1} rf_r = \frac{n}{k}(f_n + f_{n-1}) + \frac{1}{k^2}(2 - (k+2)f_n - 2f_{n-1});$$

$$\sum_{r=0}^{n-1} r^2 f_r = \frac{n^2}{k}(f_n + f_{n-1}) + \frac{1}{k^2}(2-(2n+1)((k+2)f_n + 2f_{n-1}))$$

$$-\frac{2}{k^3}((k+4)(1-f_{n-1}) - (k^2+3k+4)f_n).$$

b) Find a similar formula for $\sum_{r=0}^{n-1} r^3 f_r$, in terms of f_n, f_{n-1} and a constant term.

Editor's solution based on the proposer's solution

a) Let α and β be the solutions of the characteristic equation satisfying

$$\alpha+\beta = k, \quad \alpha\beta = -1.$$

It suffices to find, for each nonnegative integer m, the sums $\sum_{r=0}^{n-1} r^m \alpha^r$ and $\sum_{r=0}^{n-1} r^m \beta^r$. Indeed, by the Binet formula, one may write that, for r an integer

$$f_r = \frac{\alpha^r - \beta^r}{\alpha - \beta}; \quad f_{-r} = (-1)^{r+1} f_r. \tag{1}$$

The desired sums are thus expressible as follows

$$\sum_{r=0}^{n} r^m f_r = \frac{\sum_{r=0}^{n-1} r^m \alpha^r - \sum_{r=0}^{n-1} r^m \beta^r}{\alpha - \beta}. \tag{2}$$

Now note that for $z \neq 1$, we have that

$$\sum_{r=0}^{n-1} z^r = \frac{z^n - 1}{z-1}, \tag{3}$$

therefore,

$$\sum_{r=0}^{n} r z^{r-1} = \frac{d}{dz}\Big(\frac{z^n-1}{z-1}\Big) = \frac{(n-1)z - n)z^{n-1} + 1}{(z-1)^2}, \tag{4}$$

so

$$\sum_{r=0}^{n-1} r z^r = \frac{((n-1)z-n)z^n + z}{(z-1)^2}. \tag{5}$$

Note now that $\alpha \neq 1$ and $\beta \neq 1$. Indeed, if one of α or β is 1, then the other is also an integer. Since $\alpha\beta = -1$, we get that $\{\alpha, \beta\} = \{-1, 1\}$, therefore $k = \alpha + \beta = 0$, which is not allowed. Using (5) with both $z = \alpha,\ z = \beta$, subtracting the resulting expressions and using formula (2) we get

$$\sum_{r=0}^{n-1} r f_r = P_1(n)\alpha^n + P_2(n)\beta^n + P_3,$$

where $P_1(n) = ((n-1)\alpha - n)(\alpha-1)^{-2}(\alpha-\beta)^{-1}$ and $P_2(n) = -((n-1)\beta - n)(\beta-1)^{-2}(\alpha-\beta)^{-1}$ are linear polynomials in n, while $P_3 = (\alpha(1-\alpha)^2 - \beta(1-\beta)^2)(\alpha-\beta)^{-1}$ is constant. Thus, the numbers $u_n = \sum_{r=0}^{n-1} r f_r$ form a linearly recurrent sequence of order 5 whose characteristic polynomial is $(X-\alpha)^2(X-\beta)^2(X-1) = (X^2 - kX - 1)^2(X-1)$.

Let $v_n = n/k(f_n + f_{n-1}) + 1/k^2(2 - (k+2)f_n - 2f_{n-1})$ for $n = 1, 2, \ldots$. Using the Binet formula for f_n and f_{n-1}, it is easy to see that

$$v_n = Q_1(n)\alpha^n + Q_2(n)\beta^n + Q_3,$$

where $Q_1(n)$ and $Q_2(n)$ are linear polynomials in n and Q_3 is a constant. Hence, $(v_n)_{n\geq 0}$ also satisfies the 5th order linear recurrence whose characteristic polynomial is $(X^2 - kX + 1)^2(X - 1)$.

In conclusion, in order to prove that the first identity claimed at a) holds, it suffices to check the identity for the first 5 values of n. One now checks that

$$u_1 = v_1 = 0,\ u_2 = v_2 = 1,\ u_3 = v_3 = 1+2k,\ u_4 = v_4 = 3k^2+5k+1,\ u_5 = v_5 = 4k^3+3k^2+13k+1.$$

For the second identity, one can proceed in the same way. Using (4), its derivative, and (3), one shows easily that $w_n = \sum_{r=0}^{n-1} r^2 f_r$ can be represented as $R_1(n)\alpha^n + R_2(n)\beta^n + R_3$, where $R_1(n)$ and $R_2(n)$ are quadratic polynomials in n and R_3 is a constant. Hence, $(w_n)_{n\geq 0}$ satisfies the 7th order linear recurrence of characteristic polynomial $(X-\alpha)^3(X-\beta)^3(X-1) = (X^2 - kX - 1)^3(X-1)$. Now letting $(x_n)_{n\geq 0}$ be the sequence whose general term appears in the right hand side of the second identity, and using the Binet formula (1) for f_n and f_{n-1}, one sees easily that x_n is also of the form $S_1(n)\alpha^n + S_2(n)\beta^n + S_3$, where $S_1(n)$ and $S_2(n)$ are quadratic polynomials in n and S_3 is a constant. Thus, $(x_n)_{n\geq 0}$ is also a 7th order recurrent sequence satisfying the recurrence with the same characteristic equation $(X^2 - kX - 1)^3(X-1)$ as $(w_n)_{n\geq 0}$. Thus, in order to prove the second identity, it suffices to check that it indeed holds for $n = 1, \ldots, 7$, which we leave to the reader.

For b), the proposer sent in the identity

$$\sum_{r=0}^{n-1} r^3 f_r = \frac{n^3}{k}(f_n + f_{n-1}) + \frac{1}{k^2}[2 - (3n^2 + 3n + 1)((k+2)f_n + 2f_{n-1})]$$

$$-\frac{6}{k^3}[k+4-(n+1)((k^2+3k+4)f_n+(k+4)f_{n-1}]$$

$$+\frac{6}{k^4}[k^2+4k+8-(k^3+4k^2+8k+8)f_n-(k^2+4k+8)f_{n-1}],$$

which we leave to the reader to verify by the method presented above (both sides are recurrent of order 9 satisfying the same recurrence of characteristic equation $(X^2-kX-1)^4(X-1)$, so it suffices to check that it holds for the first 9 values of n).

Also solved by Kenny Davenport and the proposer.

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