ADVANCED PROBLEMS AND SOLUTIONS

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PROBLEMS PROPOSED IN THIS ISSUE

H-743 Proposed by Romeo Meštrović, Kotor, Montenegro.

Let \( p \geq 5 \) be a prime and \( q_p(2) = (2^{p-1} - 1)/p \) be the Fermat quotient of \( p \) to base 2. Prove that

\[
q_p(2) \equiv -\frac{1}{2} \sum_{k=1}^{(p-1)/2} \frac{(-3)^k}{k} \pmod{p}.
\]

H-744 Proposed by D. M. Bătinețu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.

Prove that

\[
\begin{align*}
(1) \quad & e^{n+3-L_n+2} \leq \left( \frac{1}{n} \sum_{k=1}^{n} \frac{1}{L_k} \right)^n ; \\
(2) \quad & e^{n+2-L_nL_{n+1}} \leq \left( \frac{1}{n} \sum_{k=1}^{n} \frac{1}{L_k^2} \right)^n ; \\
(3) \quad & e^{n+1-F_{n+2}} \leq \left( \frac{1}{n} \sum_{k=1}^{n} \frac{1}{F_k} \right)^n ; \\
(4) \quad & e^{n-F_nF_{n+1}} \leq \left( \frac{1}{n} \sum_{k=1}^{n} \frac{1}{F_k^2} \right)^n .
\end{align*}
\]

H-745 Proposed by Kenneth B. Davenport, SCI-Dallas, PA.

Prove that

\[
(a^2 - 1) \cos(n + 3)\theta - 2 \sqrt{a} \cos n\theta = (a - 1)^2 \cos(n + 1)\theta,
\]

where \( a \) is the real number satisfying \( a^3 = a^2 + a + 1 \) and \( \theta \) is given by \( \cos \theta = (1 - a) \sqrt{a}/2 \).

H-746 Proposed by H. Ohtsuka, Saitama, Japan.

Define the generalized Fibonomial coefficient \( \binom{n}{k}_{F;m} \) by

\[
\binom{n}{k}_{F;m} = \frac{F_{mn}F_{m(n-1)} \cdots F_{m(n-k+1)}}{F_{mk}F_{m(k-1)} \cdots F_{m}} \quad \text{for} \quad 0 \leq k \leq n
\]
with \( \binom{n}{0}_{F; m} = 1 \) and \( \binom{n}{k}_{F; m} = 0 \) (otherwise). Let \( \varepsilon_i = (-1)^{(m+1)i} \). For positive integers \( n, m \) and \( s \) prove that

\[
\sum_{i+j=2s} \varepsilon_i \binom{n}{i}_{F; m} \binom{n}{j}_{F; m} = \varepsilon_s \binom{n}{s}_{F; 2m}.
\]

SOLUTIONS

Fibonacci Numbers and Derivatives of Polynomials

H-717 Proposed by Samuel G. Moreno, Jaén, Spain, (Vol. 50, No. 2, May 2012)

Prove that if \( p \) is a polynomial such that \( p(x) > 0 \) for all \( x \in \mathbb{R} \), then

\[
\sum_{k=0}^{\text{deg}(p)} F_{k+1} y^k p^{(k)}(x) > 0 \quad \text{for all} \quad x, y \in \mathbb{R}.
\]

Solution by the proposer.

For a fixed \( y \in \mathbb{R}, \ y \neq 0 \), we consider the second-order linear differential equation with constant coefficients

\[
(I - yD - y^2D^2)q(x) = q(x) - yq'(x) - y^2q''(x) = p(x),
\]

in which \( I \) stands for the identity operator, and \( D = d/dx \) stands for the derivative. If \( \alpha \) denotes the golden ratio, the two distinct roots of the auxiliary equation of (1) are \( \lambda_1 = -\alpha/y \) and \( \lambda_2 = -(1 - \alpha)/y \). Moreover, a particular solution of (1) is

\[
q_0(x) = (I - yD - y^2D^2)^{-1} p(x) = \left( \sum_{k=0}^{\text{deg}(p)} F_{k+1} y^k D^k \right) p(x)
\]

\[
= \sum_{k=0}^{\text{deg}(p)} F_{k+1} y^k p^{(k)}(x).
\]

Thus, the general solution of (1) reads \( q(x) = q_0(x) + C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} \). Therefore, the unique polynomial solution of the differential equation considered is \( q_0 \).

Taking into account that \( p \) must be a polynomial of even degree, and also that the asymptotic behavior of \( q_0 \) is governed by \( F_1 y^0 p^{(0)}(x) = p(x) \), we observe that \( q_0 \) tends to infinity as \( |x| \) does, so there exists (at least) one absolute minimum \( m_0 \) of \( q_0 \) on \( \mathbb{R} \). Using that \( q_0'(m_0) = 0 \) and \( q_0''(m_0) \geq 0 \), and using also (1), we conclude

\[
q_0(x) \geq q_0(m_0) = p(m_0) + (yq_0'(m_0) + y^2q_0''(m_0)) = p(m_0) + y^2q_0''(m_0) > 0,
\]

for all reals \( x \).

Also solved by Paul S. Bruckman.
Inequalities with Fibonacci Numbers and Radicals

**H-718** Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 50, No. 2, May 2012)

Let \( A_{n,m} = F_{n+m}^{2n-2m-3}(F_{n+m}^4 - F_{n-m}^4) \). Prove that

1. \( \prod_{k=2m}^{2n} F_k \leq A_{n,m} \) for \( n \geq m \geq 1 \);
2. \( \prod_{k=m}^{n} F_{2k} < \sqrt{A_{n,m}} \frac{F_{2m-3}F_{2m-2}F_{2n+1}}{F_{2m-1}F_{2n-1}F_{2n}} \) for \( n \geq m \geq 2 \).

**Solution by the proposer.**

1. If \( n = m \), then \( LHS = RHS = F_{2n} \).

Let \( n > m \). We have

\[
\prod_{k=2m}^{2n} F_k = \prod_{k=2m}^{n} F_k = \prod_{j=0}^{n-m} \frac{F_{n+m-j}}{F_{n+m}} \frac{F_{n+m+j}}{F_{n+m}} = \prod_{j=0}^{n-m} \left( 1 - \frac{(-1)^{n+m-j}F_j^2}{F_{n+m}} \right) \quad \text{(By Catalan's Identity)}
\]

If \( n - m \) is odd,

\[
\prod_{j=0}^{n-m} \left( 1 - \frac{(-1)^{n+m-j}F_j^2}{F_{n+m}} \right) = \prod_{r=0}^{(n-m)/2} \left( 1 + \frac{F_{2r}^2}{F_{n+m}} \right) \left( 1 - \frac{F_{2r+1}^2}{F_{n+m}} \right) \leq \prod_{r=0}^{(n-m)/2} \left( 1 - \frac{F_{2r+1}^2}{F_{n+m}^4} \right) \leq 1 - \frac{F_{n-m}^4}{F_{n+m}^4}.
\]

If \( n - m \) is even,

\[
\prod_{j=0}^{n-m} \left( 1 - \frac{(-1)^{n+m-j}F_j^2}{F_{n+m}} \right) = \prod_{r=0}^{(n-m)/2} \left( 1 + \frac{F_{2r-1}^2}{F_{n+m}} \right) \left( 1 - \frac{F_{2r}^2}{F_{n+m}} \right) \leq \prod_{r=0}^{(n-m)/2} \left( 1 - \frac{F_{2r}^2}{F_{n+m}^4} \right) \leq 1 - \frac{F_{n-m}^4}{F_{n+m}^4}.
\]
Therefore, we obtain
\[ \prod_{k=2m}^{2n} F_k \leq \frac{F^n_{2m-2m+1}}{F^n_{n+m}} \left( 1 - \frac{F^n_{m-n}}{F^n_{n+m}} \right) = A_{n,m}. \]

(2) First, we have \( F_{t-2}F_{t-1}F_{t+1}F_{t+2} < F_t^4 \) by the Gelin–Cesàre Identity. Therefore, for \( t \geq 3 \), we have
\[ \frac{F_{t-1}F_{t+1}}{F_{t-2}F_{t+2}} < \frac{F^2_t}{F^2_{t}}. \] (1)

Let \( n \geq m \geq 2 \). We have
\[ \prod_{k=m}^{n} \frac{F^2_{2k}}{F^2_{2k-1}} = \frac{F_{2n}}{F_{2m-2}} \prod_{k=m}^{n} \frac{F_{2k-2}F_{2k}}{F^2_{2k-1}} < \frac{F_{2n}}{F_{2m-2}} \prod_{k=m}^{n} \frac{F^2_{2k-1}}{F_{2k-3}F_{2k+1}} \] (by (1))
\[ = \frac{F_{2m-1}F_{2n-1}F_{2n}}{F_{2m-3}F_{2m-2}F_{2n+1}}. \]

Thus, we have
\[ \prod_{k=m}^{n} F_{2k} < \sqrt{\frac{F_{2m-1}F_{2n-1}F_{2n}}{F_{2m-3}F_{2m-2}F_{2n+1}}} \prod_{k=m}^{n} F_{2k-1}. \]

Multiplying both sides of this inequality by \( \prod_{k=m}^{n} F_{2k} \), we get
\[ \prod_{k=m}^{n} F^2_{2k} < \sqrt{\frac{F_{2m-1}F_{2n-1}F_{2n}}{F_{2m-3}F_{2m-2}F_{2n+1}}} \prod_{k=m}^{n} F_{2k-1}F_{2k}. \]

Here, we have
\[ \prod_{k=m}^{n} F_{2k-1}F_{2k} = \prod_{k=2m-1}^{2n} F_k = \prod_{k=2m}^{2n} F_k \leq F_{2m-1}A_{n,m} \] (by (1)).

Thus, we have
\[ \prod_{k=m}^{n} F^2_{2k} < A_{n,m} \sqrt{\frac{F^3_{2m-1}F_{2n-1}F_{2n}}{F_{2m-3}F_{2m-2}F_{2n+1}}}. \]

which leads to the desired inequality.

**Note.** We obtain the following inequality in the same manner as (2):
\[ \prod_{k=m}^{n} F_{2k-1} < \sqrt{A_{n,m}} \sqrt{\frac{F^3_{2m-1}F_{2n-1}F_{2n}}{F_{2m-2}F_{2n+1}F_{2n+2}}} \] (for \( n \geq m \geq 2 \)).

Also solved by Paul S. Bruckman and Dmitry Fleischman.
Alternating Sums of High Powers of Fibonacci Numbers

Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Let \( T_j(n) = (-1)^{n(j+1)}(F_nF_{n+1})^j \). Given a positive integer \( m \) prove that there are rational numbers \( \lambda_1, \ldots, \lambda_m \) such that

\[
\sum_{k=1}^{n} (-1)^{k(m+1)} F_k^{2m} = \sum_{j=1}^{m} \lambda_j T_j(n).
\]

Show the identities

(1) \[
\sum_{k=1}^{n} (-1)^k F_k^4 = -\frac{2}{3} T_1(n) + \frac{1}{3} T_2(n);
\]

(2) \[
\sum_{k=1}^{n} F_k^6 = \frac{1}{2} T_1(n) - \frac{1}{4} T_2(n) + \frac{1}{4} T_3(n);
\]

(3) \[
\sum_{k=1}^{n} (-1)^k F_k^8 = -\frac{8}{21} T_1(n) + \frac{4}{21} T_2(n) - \frac{2}{7} T_3(n) + \frac{1}{7} T_4(n).
\]

Solution by Harris Kwong, SUNY Fredonia, NY.

Lemma. For any integer \( i \geq 1 \), there exist rational numbers \( a_{i,\ell} \) such that

\[
F_k^i = F_{k+1}^i + (-1)^i F_{k-1}^i + \sum_{\ell=0}^{[i/2]} a_{i,\ell} (-1)^{i\ell} F_k^{i-2\ell}.
\]

Equivalently, we can write

\[
F_{k+1}^i + (-1)^i F_{k-1}^i = \sum_{\ell=0}^{[i/2]} b_{i,\ell} (-1)^{i\ell} F_k^{i-2\ell}
\]

for some rational numbers \( b_{i,\ell} \).

Proof. Induct on \( i \). The result is obviously true when \( i = 1 \), because \( F_{k+1} - F_{k-1} = F_k \). For \( i \geq 2 \),

\[
F_k^i = (F_{k+1} - F_{k-1})^i = F_{k+1}^i + (-1)^i F_{k-1}^i + \sum_{r=1}^{i-1} (-1)^r \binom{i}{r} F_{k+1}^{i-r} F_{k-1}^r.
\]

When \( i \) is even, Casini’s identity \( F_{k+1}F_{k-1} = F_k^2 + (-1)^k \) implies that the middle term in the summation, where \( r = i/2 \), is

\[
(-1)^{i/2} \binom{i}{i/2} (F_{k+1}F_{k-1})^{i/2} = (-1)^{i/2} \binom{i}{i/2} [F_k^2 + (-1)^k]^{i/2}
\]

\[
= (-1)^{i/2} \binom{i}{i/2} \sum_{\ell=0}^{i/2} \binom{i/2}{\ell} F_k^{2(i/2-\ell)} (-1)^{i\ell}
\]

\[
= \sum_{\ell=0}^{i/2} c_{i/2,\ell} (-1)^{i\ell} F_k^{i-2\ell},
\]

\[
\sum_{k=1}^{n} (-1)^k F_k^4 = -\frac{2}{3} T_1(n) + \frac{1}{3} T_2(n);
\]

\[
\sum_{k=1}^{n} F_k^6 = \frac{1}{2} T_1(n) - \frac{1}{4} T_2(n) + \frac{1}{4} T_3(n);
\]

\[
\sum_{k=1}^{n} (-1)^k F_k^8 = -\frac{8}{21} T_1(n) + \frac{4}{21} T_2(n) - \frac{2}{7} T_3(n) + \frac{1}{7} T_4(n).
\]
Thus, 3 leads to

\[
\sum_{i=0}^{n} \left( \sum_{r=0}^{i} \binom{i}{r} \right)^2 (i/2) (i/2) = (1) \sum_{i=0}^{n} \left( \sum_{r=0}^{i} \binom{i}{r} \right)^2 (i/2) (i/2).
\]

In general, for 1 \( \leq r \leq |(i-1)/2| \), due to symmetry, we can group the \( r \)th term with the \((i-r)\)th term; and it follows from the induction hypothesis that

\[
(-1)^{r} \binom{i}{r} F_{k+1}^{i-r} F_{k-1}^{r} + (-1)^{i-r} \binom{i}{i-r} F_{k+1}^{i-r} F_{k-1}^{r}
\]

\[
= (-1)^{r} \binom{i}{r} (F_{k+1}F_{k-1})^{r}[F_{k+1}^{-2r} + (-1)^{-2r} F_{k-1}^{-2r}]
\]

\[
= (-1)^{r} \binom{i}{r} (F_{k}^{2} + (-1)^{k})^{r}[F_{k+1}^{-2r} + (-1)^{-2r} F_{k-1}^{-2r}]
\]

\[
= (-1)^{r} \sum_{r=0}^{i} \binom{r}{r} F_{k}^{2(r-s)} (i) \left( \sum_{t=0}^{2(i-2r)} b_{r-t, t} (1)^{kt} F_{k}^{-2r-2t} \right)
\]

\[
= \sum_{\ell=0}^{\lfloor i/2 \rfloor} c_{r, \ell} (-1)^{k} \ell F_{k}^{-2\ell}
\]

where \( c_{r, \ell} = \sum_{s=0}^{\lfloor i/2 \rfloor} (i) F_{s} \) is a rational number. The result follows immediately. \( \square \)

We now prove the original problem. The case of \( m = 1 \) is valid:

\[
\sum_{k=1}^{n} F_{k}^{2} = F_{n}F_{n+1} = T_{1}(n).
\]

Since \( F_{k}^{2m} = F_{k}^{m} \cdot F_{k}^{m} \), the lemma asserts that

\[
\sum_{k=1}^{n} (-1)^{k(m+1)} F_{k}^{2m} = \sum_{k=1}^{n} (-1)^{k(m+1)} F_{k}^{m} \left( F_{k+1}^{m} + (-1)^{m} F_{k-1}^{m} + \sum_{\ell=0}^{[m/2]} a_{m, \ell} (-1)^{\ell} F_{k}^{m-2\ell} \right)
\]

\[
= (-1)^{n(m+1)} F_{n} F_{n+1}^{m} + \sum_{\ell=0}^{[m/2]} a_{m, \ell} \sum_{k=1}^{n} (-1)^{k(m-\ell+1)} F_{k}^{2(m-\ell)}.
\]

Solving for \( \sum_{k=1}^{n} (-1)^{k(m+1)} F_{k}^{2m} \) yields the desired result from induction.

In practice, it is easier to compute the coefficients \( \lambda_{j} \) directly. For example, when \( m = 2 \),

\[
F_{k}^{2} = (F_{k+1} - F_{k-1})^{2} = F_{k+1}^{2} + F_{k-1}^{2} - 2F_{k+1}F_{k-1} = F_{k+1}^{2} + F_{k-1}^{2} - 2F_{k}^{2} - 2(-1)^{k}.
\]

This leads to

\[
\sum_{k=1}^{n} (-1)^{k} F_{k}^{4} = \sum_{k=1}^{n} (-1)^{k} F_{k}^{2} (F_{k+1}^{2} + F_{k-1}^{2}) - 2 \sum_{k=1}^{n} (-1)^{k} F_{k}^{4} - 2 \sum_{k=1}^{n} F_{k}^{2}
\]

\[
= (-1)^{n} F_{n}^{2} F_{n+1}^{2} - 2 \sum_{k=1}^{n} (-1)^{k} F_{k}^{4} - 2 T_{1}(n).
\]

Thus, \( 3 \sum_{k=1}^{n} (-1)^{k} F_{k}^{4} = T_{2}(n) - 2T_{1}(n) \), which proves (1).

In a similar manner, we find

\[
F_{k}^{3} = F_{k+1}^{3} - F_{k-1}^{3} - 3F_{k+1}F_{k-1}(F_{k+1} - F_{k-1}) = F_{k+1}^{3} - F_{k-1}^{3} - 3[F_{k}^{2} + (-1)^{k}]F_{k}.
\]
Hence,
\[
\sum_{k=1}^{n} F_k^6 = \sum_{k=1}^{n} F_k^3 (F_{k+1}^3 - F_{k-1}^3) - 3 \sum_{k=1}^{n} F_k^6 - 3 \sum_{k=1}^{n} (-1)^k F_k^4
\]
\[
= F_n^3 F_{n+1}^3 - 3 \sum_{k=1}^{n} F_k^6 - 3 \left( \frac{1}{3} T_2(n) - \frac{2}{3} T_1(n) \right).
\]
This yields
\[
4 \sum_{k=1}^{n} F_k^6 = T_3(n) - T_2(n) + 2T_1(n),
\]
thereby proving (2).

The case of \(m = 4\) is slightly more complicated. First we obtain
\[
F_k^4 = F_{k+1}^4 + F_{k-1}^4 - 4F_{k+1}F_{k-1}(F_{k+1}^2 + F_{k-1}^2) + 6F_{k+1}^2 F_{k-1}^2
\]
\[
= F_{k+1}^4 + F_{k-1}^4 - 4F_{k+1}F_{k-1}[(F_{k+1} - F_{k-1})^2 + 2F_{k+1}F_{k-1}] + 6F_{k+1}^2 F_{k-1}^2
\]
\[
= F_{k+1}^4 + F_{k-1}^4 - 4F_{k+1}F_{k-1} - 2F_{k+1}^2 F_{k-1}
\]
\[
= F_{k+1}^4 + F_{k-1}^4 - 4[F_k^2 + (-1)^k]F_k^2 - 2[F_k^2 + (-1)^k]^2
\]
\[
= F_{k+1}^4 + F_{k-1}^4 - 6F_k^4 - 8(-1)^k F_k^2 - 2.
\]
Therefore,
\[
\sum_{k=1}^{n} (-1)^k F_k^8 = \sum_{k=1}^{n} (-1)^k F_k^4 (F_{k+1}^4 + F_{k-1}^4) - 6 \sum_{k=1}^{n} (-1)^k F_k^8 - 8 \sum_{k=1}^{n} F_k^6 - 2 \sum_{k=1}^{n} (-1)^k F_k^4.
\]
We conclude that
\[
\sum_{k=1}^{n} (-1)^n F_k^8 = \frac{1}{7} \left[ T_4(n) - 8 \left( \frac{1}{4} T_3(n) - \frac{1}{4} T_2(n) + \frac{1}{2} T_1(n) \right) \right] - 2 \left( \frac{1}{3} T_2(n) - \frac{2}{3} T_1(n) \right)
\]
\[
= \frac{1}{7} T_4(n) + \frac{2}{7} T_3(n) + \frac{4}{21} T_2(n) - \frac{8}{21} T_1(n),
\]
which establishes (3).

Also solved by Paul S. Bruckman, Kenneth B. Davenport, Dmitry Fleischman and Zbigniew Jakubczyk.

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