Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, SCHOOL OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG X3, WITS 2050, JOHANNESBURG, SOUTH AFRICA or by e-mail at the address florian.luca@wits.ac.za as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE


For any nonnegative integers \(n, m, l\) prove that

\[
\sum_{k=0}^{n} \binom{n}{k}^2 \sum_{i \geq 0} \binom{2k}{i} \binom{2n-2k}{m-i} (-1)^{m-i} = \begin{cases} \binom{2l}{l} \binom{2n-2l}{n-l} & \text{if } m = 2l; \\ 0 & \text{if } m = 2l+1. \end{cases}
\]

H-778 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Prove that

\[
\sum_{n=1}^{\infty} \frac{1}{(-\sqrt{5})^n F_2 F_4 F_8 \cdots F_{2n}} = \frac{\sqrt{5} - 3}{2}.
\]

H-779 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Let \(\binom{n}{k}_F\) denote the Fibonomial coefficient. For integers \(n \geq 1\) and \(r \neq 0\) with \(n + r \neq 0\), prove that

\[
\sum_{k=0}^{n} (-1)^{k+1/2} F_{k+r} \left( \frac{F_r}{F_{n+r}} \right)^k \binom{n}{k}_F = 0.
\]

H-780 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Given real numbers \(r\) and \(t > 0\) and an integer \(n \geq 0\) find a closed form expression for the sum:

\[
\sum_{k=0}^{n} \frac{1}{f_k (L_{2k}^r + t)(L_{2k+1}^r + t) \cdots (L_{2n}^r + t)},
\]

where \(f_0 = t/(t+1)\) and \(f_k = F_{2k+1}^r\) for \(k \geq 1\).

H-781 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Find a closed form expression for the sums:
HYPERGEOMETRIC TEMPLATE

(i) \[ \sum_{k=1}^{n} (L_{2^k} \pm \sqrt{5})(L_{2^{k+1}} \pm \sqrt{5}) \cdots (L_{2^n} \pm \sqrt{5}) \] for \( n \geq 1 \);

(ii) \[ \sum_{k=m+1}^{n} (L_{2^k} \pm L_{2^m})(L_{2^{k+1}} \pm L_{2^m}) \cdots (L_{2^n} \pm L_{2^m}) \] for \( n > m \geq 1 \).

SOLUTIONS

Sums of Products of Fibonomials, Fibonacci and Lucas Numbers

H-747 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 52, No. 1, February 2014)

Let \( \binom{n}{k}_F \) denote the Fibonomial coefficient. For positive integer \( n \), find closed form expressions for the sums:

(i) \[ \sum_{k=0}^{n-1} (-1)^k F^2_{2k} (L_{k+1} L_{k+2} \cdots L_n)^2 \binom{2k}{k}_F \];

(ii) \[ \sum_{k=0}^{n-1} (-1)^k F^4_{4k+1} (L_{k+1} L_{k+2} \cdots L_n)^4 \binom{2k}{k}_F^2 \].

Solution by Harris Kwong, SUNY, Fredonia.

Denote the given sums \( S_n \) and \( T_n \), respectively. We shall use induction to prove that

\[ S_n = \frac{(-1)^{n-1} F_{2n} F_{2n-2}}{2} \binom{2n}{n}_F, \] and \( T_n = (-1)^{n-1} F^2_{2n} \binom{2n}{n}_F \).

(i) The definition states that \( S_1 = F^2_0 L^2_1 \binom{0}{0}_F = 0 \), and the formula says \( S_1 = \frac{1}{2} F^2_2 F^2_0 \binom{2}{1}_F = 0 \). This verifies the base case \( n = 1 \). Assume the formula holds for some integer \( n \geq 1 \). Then, because \( L_{n+1} F_{n+1} = F_{2n+2} \), and \( 2F_{2n} - F_{2n-2} = F_{2n-1} \), we obtain

\[
S_{n+1} = L_{n+1}^2 \left[ S_n + (-1)^n F^2_{2n} \binom{2n}{n}_F \right]
\]
\[
= L_{n+1}^2 \left[ \frac{(-1)^{n-1} F_{2n} F_{2n-2}}{2} + (-1)^n F^2_{2n} \right] \binom{2n}{n}_F
\]
\[
= \frac{(-1)^n L^2_{n+1} F_{2n} (2F_{2n} - F_{2n-2})}{2 F_{2n+2} F_{2n+1}} \binom{2n+2}{n+1}_F
\]
\[
= \frac{(-1)^n F^2_{2n+2} F_{2n}}{2} \binom{2n+2}{n+1}_F,
\]

thereby completing the induction.

(ii) The definition states that \( T_1 = F_1 L^2_1 \binom{0}{0}_F = 1 \), which agrees with the formula \( T_1 = F^2_2 \binom{2}{1}_F \). Noting that \( L_{n+1} F_{n+1} = F_{2n+2} \) and \( F_{4n+1} - F^2_{2n} = F^2_{2n+1} \), we establish the inductive step as
follows:

\[ T_{n+1} = L_{n+1}^4 \left[ T_n + (-1)^n F_{4n+1} \left( \frac{2n}{n} \right)_F \right] = L_{n+1}^4 \left[ ( -1)^{n-1} F_{2n}^2 + (-1)^n F_{4n+1} \right] \left( \frac{2n}{n} \right)_F = ( -1)^n L_{n+1}^4 \left( F_{4n+1} - F_{2n}^2 \right) \left( \frac{F_{n+1}^2}{F_{2n+2} F_{2n+1}} \right)^2 \left( \frac{2n+2}{n+1} \right)_F = ( -1)^n F_{2n+2}^2 \left( \frac{2n+2}{n+1} \right)_F. \]

Also solved by the proposer.

**Some Nesbitt Type Inequalities With Fibonacci and Lucas Numbers**

\textbf{H-748} Proposed by D. M. Bătinețu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.
(Vol. 52, No. 1, February 2014)

Let \( x_k = L_k, y_k = F_k, k = 1, \ldots, m, x_{m+1} = x_1, y_{m+1} = y_1. \) Prove that:

\[
\frac{2}{F_{n+2}} + \sum_{k=1}^m \frac{x_k^3}{F_{n+1} x_k + F_n x_{k+1}} \geq \frac{L_m L_{m+1}}{F_{n+2}},
\]

\[
\sum_{k=1}^m \frac{y_k^3}{L_m y_k + L_{m+1} y_{k+1}} \geq \frac{F_m F_{m+1}}{L_{m+2}}
\]

for every positive integer \( n. \)

**Solution by Ángel Plaza, Gran Canaria, Spain.**

Both inequalities are consequence of the following Nesbitt type more general inequality where the left-hand side sum is cyclic

\[
\sum_{k=1}^m \frac{x_k^3}{a x_k + b x_{k+1}} \geq \frac{\sum_{k=1}^m x_k^2}{a + b}.
\]

Then the left-hand side of the fist equation, \( LHS \) is

\[
LHS \geq \frac{2}{F_{n+2}} + \sum_{k=1}^m \frac{x_k^2}{F_n + F_{n+1}} = \frac{2 + L_m L_{m+1}}{F_{n+2}} = \frac{L_m L_{m+1}}{F_{n+2}},
\]

where we have used that \( \sum_{k=1}^m L_k^2 = L_m L_{m+1} - 2. \)

The second inequality is proved in the same way by now using that \( \sum_{k=1}^m F_k^2 = F_m F_{m+1}. \)

Also solved by Dmitry Fleischman and the proposers.
Identities With Sums of Ratios of Fibonacci Numbers and Products of Them

**H-749** Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 52, No. 1, February 2014)

Let \(a, b, c\) and \(d\) be odd positive integers. If \(a + b = c + d\), prove that

\[
\sum_{k=1}^{b} \frac{F_a}{F_k F_{k+a}} + \sum_{k=1}^{a} \frac{F_b}{F_k F_{k+b}} = \sum_{k=1}^{d} \frac{F_c}{F_k F_{k+c}} + \sum_{k=1}^{c} \frac{F_d}{F_k F_{k+d}}.
\]

**Solution by Ángel Plaza, Gran Canaria, Spain.**

We use the following reduction formula [1, Theorem 6]

\[
\sum_{k=1}^{N} \frac{1}{F_k F_{k+a}} = \frac{1}{F_a} \sum_{k=1}^{[a/2]} \left( \frac{1}{F_{b+2k} F_{b+2k+1}} - \frac{1}{F_{2k} F_{2k+1}} \right) + \mathbb{K}_b
\]

with \(\mathbb{K}_N = \sum_{k=1}^{N} \frac{1}{F_k F_{k+1}}\). Therefore, the \(LHS\) and the \(RHS\) of the proposed identity are, respectively

\[
LHS = \sum_{k=1}^{[a/2]} \left( \frac{1}{F_{b+2k} F_{b+2k+1}} - \frac{1}{F_{2k} F_{2k+1}} \right) + \mathbb{K}_b + \sum_{k=1}^{[b/2]} \left( \frac{1}{F_{a+2k} F_{a+2k+1}} - \frac{1}{F_{2k} F_{2k+1}} \right) + \mathbb{K}_a
\]

\[
RHS = \sum_{k=1}^{[c/2]} \left( \frac{1}{F_{d+2k} F_{d+2k+1}} - \frac{1}{F_{2k} F_{2k+1}} \right) + \mathbb{K}_d + \sum_{k=1}^{[d/2]} \left( \frac{1}{F_{c+2k} F_{c+2k+1}} - \frac{1}{F_{2k} F_{2k+1}} \right) + \mathbb{K}_c.
\]

Since \(a, b, c\) and \(d\) are odd positive integers with \(a + b = c + d\) we may assume that \(a = 2\alpha + 1\), \(c = a + 2m\), \(b = 2\beta + 1\) and \(d = b - 2m\). Then \([a/2] = \alpha\), \([b/2] = \beta\), \([c/2] = \alpha + m\), and \([d/2] = \beta - m\). Previous expressions for \(LHS\) and \(RHS\) are now

\[
LHS = \sum_{k=1}^{\alpha} \left( \frac{1}{F_{b+2k} F_{b+2k+1}} - \frac{1}{F_{2k} F_{2k+1}} \right) + \sum_{k=1}^{b} \frac{1}{F_k F_{k+1}}
\]

\[
+ \sum_{k=1}^{\beta} \left( \frac{1}{F_{a+2k} F_{a+2k+1}} - \frac{1}{F_{2k} F_{2k+1}} \right) + \sum_{k=1}^{a} \frac{1}{F_k F_{k+1}}
\]

\[
RHS = \sum_{k=1}^{\alpha+m} \left( \frac{1}{F_{b-2m+2k} F_{b-2m+2k+1}} - \frac{1}{F_{2k} F_{2k+1}} \right) + \sum_{k=1}^{b-2m} \frac{1}{F_k F_{k+1}}
\]

\[
+ \sum_{k=1}^{\beta-m} \left( \frac{1}{F_{a+2m+2k} F_{a+2m+2k+1}} - \frac{1}{F_{2k} F_{2k+1}} \right) + \sum_{k=1}^{a+2m} \frac{1}{F_k F_{k+1}}.
\]
where after cancelling common terms we have to prove that \( LHS^* = RHS^* \):

\[
LHS^* = \sum_{k=b-2m+1}^{b} \frac{1}{F_k F_{k+1}} + \sum_{k=1}^{m} \frac{1}{F_{a+2k} F_{a+2k+1}}
\]

\[
RHS^* = \sum_{k=1}^{m} \frac{1}{F_{b-2m+2k} F_{b-2m+2k+1}} + \sum_{k=a+1}^{a+2m} \frac{1}{F_k F_{k+1}},
\]

which are clearly the same.

**References**


Also solved by the proposer.

**Identities With Generalized Tribonacci Recurrences**

**H-750** Proposed by Hideyuki Ohtsuka, Saitama, Japan.  
(Vol. 52, No. 1, February 2014)

Generalized Tribonacci sequences \( \{R_n\} \) and \( \{S_n\} \) are defined by

\[
R_{n+3} = pR_{n+2} + qR_{n+1} + rR_n \quad \text{(for } n \geq 0);\]

\[
S_{n+3} = pS_{n+2} + qS_{n+1} + rS_n \quad \text{(for } n \geq 0),
\]

with arbitrary \( p, q, r, R_0, R_1, R_2, S_0, S_1, S_2 \). For positive integers \( a, b, c, d \) such that \( a+b = c+d \), prove that

\[
R_{a+3}S_{b+3} + qR_{a+2}S_{b+2} + prR_{a+1}S_{b+1} - r^2R_aS_b = R_{c+3}S_{d+3} + qR_{c+2}S_{d+2} + prR_{c+1}S_{d+1} - r^2R_cS_d.
\]

**Solution by the proposer.**

We have

\[
R_{a+3}S_{b+3} - R_{a+2}S_{b+4} + qR_{a+2}S_{b+2} - qR_{a+1}S_{b+3}
\]

\[
= S_{b+3}(R_{a+3} - qR_{a+1}) - R_{a+2}(S_{b+4} - qS_{b+2})
\]

\[
= S_{b+3}(pR_{a+2} + rR_a) - R_{a+2}(pS_{b+3} + rS_{b+1})
\]

\[
= rS_{b+3}R_a - rR_{a+2}S_{b+1}
\]

\[
= r(pS_{b+2} + qS_{b+1} + rS_b)R_a - r(pR_{a+1} + qR_a + rR_{a-1})S_{b+1}
\]

\[
= r^2R_aS_b - r^2R_{a-1}S_{b+1} - prR_{a+1}S_{b+1} + prR_aS_{b+2}.
\]

Thus,

\[
R_{a+3}S_{b+3} + qR_{a+2}S_{b+2} + prR_{a+1}S_{b+1} - r^2R_aS_b
\]

\[
= R_{a+2}S_{b+4} + qR_{a+1}S_{b+3} + prR_aS_{b+2} - r^2R_{a-1}S_{b+1}.
\]

Letting \( A_{a,b} = R_{a+3}S_{b+3} + qR_{a+2}S_{b+2} + prR_{a+1}S_{b+1} - r^2R_aS_b \), we have \( A_{a,b} = A_{a-1,b+1} \). Using this identity repeatedly,

\[
\cdots = A_{a+2,b-2} = A_{a+1,b-1} = A_{a,b} = A_{a-1,b+1} = A_{a-2,b+2} = \cdots
\]

Thus, we have \( A_{a,b} = A_{a-j,b+j} \). That is, \( A_{a,b} = A_{c,d} \) for \( a + b = c + d \). Therefore we obtain the desired identity.
An Inequality With Sums of Binomial Coefficients and Fibonacci Numbers

**H-751** Proposed by D. M. Bătinețu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania. 
(Vol. 52, No. 2, May 2014)

Prove that

\[
\left( \frac{F_{2n}^{m+1} + \binom{2n+1}{1} F_{m+1}^{2n-1} F_m + \cdots + \binom{2n+1}{n} F_{m+1}^{2n-1} F_m F_m}{F_m} \right)^{p+1} \\
+ \left( \frac{\binom{2n+1}{n+1} F_{m+1}^n F_m + \cdots + \binom{2n+1}{2n} F_{m+1}^{2n-1} F_m F_m}{F_{m+1}} \right)^{p+1} \geq \frac{1}{2^p} \left( \frac{F_{m+2}^{2n+1}}{F_m F_{m+1}} \right)^{p+1}
\]

holds for any \( p \geq 0 \) and positive integers \( m \) and \( n \), and that the same inequality holds with all the \( F \)'s replaced by \( L \)'s.

**Solution by Harris Kwong, SUNY, Fredonia.**

The left-hand side of the inequality is in the form of \( y_1^{p+1} + y_2^{p+1} \), where

\[
y_1 = \frac{F_{2n}^{m+1} + \binom{2n+1}{1} F_{m+1}^{2n-1} F_m + \cdots + \binom{2n+1}{n} F_{m+1}^{2n-1} F_m}{F_m}, \\
[3pt] y_2 = \frac{\binom{2n+1}{n+1} F_{m+1}^n F_m + \cdots + \binom{2n+1}{2n} F_{m+1}^{2n-1} F_m F_m}{F_{m+1}}.
\]

Notice that

\[
F_m F_{m+1} (y_1 + y_2) = (F_{m+1} + F_m)^{2n+1} = F_{m+2}^{2n+1}.
\]

For any positive numbers \( x_1, x_2, \ldots, x_k \), it is well-known that

\[
f(r) = \left( \frac{x_1^r + x_2^r + \cdots + x_k^r}{k} \right)^{\frac{1}{r}}
\]

is an increasing function of \( r \). Hence,

\[
\left( \frac{y_1^{p+1} + y_2^{p+1}}{2} \right)^{\frac{1}{p+1}} \geq \frac{y_1 + y_2}{2} = \frac{1}{2} \left( \frac{F_{m+2}^{2n+1}}{F_m F_{m+1}} \right),
\]

from which the desired inequality follows immediately, and it is clear that it also holds when all the \( F \)'s are replaced by \( L \)'s.

**Also solved by Dmitry Fleischman, Kenneth B. Davenport, Zbigniew Jakubczyk, Ángel Plaza, and the proposers.**

**Errata:** In the statement of Advanced Problem **H-751**, there was an additional term \( F_{m+1}^{2n} \) in the numerator of the second fraction is the left–hand side of the inequality to be proven. The present solution takes this into account.