ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
FLORIAN LUCA

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, SCHOOL OF MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG X3, WITS 2050, JOHANNESBURG, SOUTH AFRICA or by e-mail at the address florian.luca@wits.ac.za as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-777 Proposed by Kiyoshi Kawazu, Izumi Kubo and Toshio Nakata, Japan.

For any nonnegative integers $n,\ m,\ l$ prove that

$$\sum_{k=0}^{n}\binom{n}{k}^2\sum_{i\geq 0}\binom{2k}{i}\binom{2n-2k}{m-i}(-1)^{m-i}=\begin{cases}\binom{2l}{l}\binom{2n-2l}{n-l} & \text{if} \quad m=2l;\\ 0 & \text{if} \quad m=2l+1.\end{cases}$$

H-778 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Prove that

$$\sum_{n=1}^{\infty}\frac{1}{(-\sqrt{5})^n F_2F_4F_8\cdots F_{2^n}}=\frac{\sqrt{5}-3}{2}.$$

H-779 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Let $\binom{n}{k}_F$ denote the Fibonomial coefficient. For integers $n\geq 1$ and $r\neq 0$ with $n+r\neq 0$, prove that

$$\sum_{k=0}^{n}(-1)^{k(k+1)/2}F_{k+r}\left(\frac{F_r}{F_{n+r}}\right)^k\binom{n}{k}_F=0.$$

H-780 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Given real numbers r and $t>0$ and an integer $n\geq 0$ find a closed form expression for the sum:

$$\sum_{k=0}^{n}\frac{1}{f_k(L_{2^k}^r+t)(L_{2^{k+1}}^r+t)\cdots(L_{2^n}^r+t)},$$

where $f_0=t/(t+1)$ and $f_k=F_{2^{k+1}}^r$ for $k\geq 1$.

H-781 Proposed by Hideyuki Ohtsuka, Saitama, Japan.

Find a closed form expression for the sums:

(i) $\displaystyle\sum_{k=1}^{n}(L_{2^k} \pm \sqrt{5})(L_{2^{k+1}} \pm \sqrt{5})\cdots(L_{2^n} \pm \sqrt{5})$ for $n \geq 1$;

(ii) $\displaystyle\sum_{k=m+1}^{n}(L_{2^k} \pm L_{2^m})(L_{2^{k+1}} \pm L_{2^m})\cdots(L_{2^n} \pm L_{2^m})$ for $n > m \geq 1$.

SOLUTIONS

Sums of Products of Fibonomials, Fibonacci and Lucas Numbers

H-747 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 52, No. 1, February 2014)

Let $\binom{n}{k}_F$ denote the Fibonomial coefficient. For positive integer n, find closed form expressions for the sums:

(i) $\sum_{k=0}^{n-1}(-1)^k F_{2k}^2 \left(L_{k+1}L_{k+2}\cdots L_n\right)^2 \binom{2k}{k}_F$;

(ii) $\sum_{k=0}^{n-1}(-1)^k F_{4k+1} \left(L_{k+1}L_{k+2}\cdots L_n\right)^4 \binom{2k}{k}_F^2$.

Solution by Harris Kwong, SUNY, Fredonia.

Denote the given sums S_n and T_n, respectively. We shall use induction to prove that

$$S_n = \frac{(-1)^{n-1}F_{2n}F_{2n-2}}{2}\binom{2n}{n}_F, \qquad \text{and} \qquad T_n = (-1)^{n-1}F_{2n}^2\binom{2n}{n}_F^2.$$

(i) The definition states that $S_1 = F_0^2 L_1^2 \binom{0}{0}_F = 0$, and the formula says $S_1 = \frac{1}{2}F_2F_0\binom{2}{1}_F = 0$. This verifies the base case $n = 1$. Assume the formula holds for some integer $n \geq 1$. Then, because $L_{n+1}F_{n+1} = F_{2n+2}$, and $2F_{2n} - F_{2n-2} = F_{2n+1}$, we obtain

$$\begin{aligned}
S_{n+1} &= L_{n+1}^2\left[S_n + (-1)^n F_{2n}^2\binom{2n}{n}_F\right] \\
&= L_{n+1}^2\left[\frac{(-1)^{n-1}F_{2n}F_{2n-2}}{2} + (-1)^n F_{2n}^2\right]\binom{2n}{n}_F \\
&= \frac{(-1)^n L_{n+1}^2 F_{2n}(2F_{2n} - F_{2n-2})}{2}\cdot\frac{F_{n+1}^2}{F_{2n+2}F_{2n+1}}\binom{2n+2}{n+1}_F \\
&= \frac{(-1)^n F_{2n+2}F_{2n}}{2}\binom{2n+2}{n+1}_F,
\end{aligned}$$

thereby completing the induction.

(ii) The definition states that $T_1 = F_1 L_1^4 \binom{0}{0}_F^2 = 1$, which agrees with the formula $T_1 = F_2^2\binom{2}{1}_F^2$. Noting that $L_{n+1}F_{n+1} = F_{2n+2}$ and $F_{4n+1} - F_{2n}^2 = F_{2n+1}^2$, we establish the inductive step as follows:

$$\begin{aligned}T_{n+1} &= L_{n+1}^4\left[T_n + (-1)^n F_{4n+1}\binom{2n}{n}_F^2\right]\\ &= L_{n+1}^4\left[(-1)^{n-1}F_{2n}^2 + (-1)^n F_{4n+1}\right]\binom{2n}{n}_F^2\\ &= (-1)^n L_{n+1}^4\big(F_{4n+1} - F_{2n}^2\big)\left(\frac{F_{n+1}^2}{F_{2n+2}F_{2n+1}}\right)^2\binom{2n+2}{n+1}_F^2\\ &= (-1)^n F_{2n+2}^2\binom{2n+2}{n+1}_F^2.\end{aligned}$$

Also solved by the proposer.

Some Nesbitt Type Inequalities With Fibonacci and Lucas Numbers

H-748 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.
(Vol. 52, No. 1, February 2014)

Let $x_k = L_k$, $y_k = F_k$, $k = 1, \ldots, m$, $x_{m+1} = x_1,\ y_{m+1} = y_1$. Prove that:

$$\frac{2}{F_{n+2}} + \sum_{k=1}^m \frac{x_k^3}{F_{n+1}x_k + F_n x_{k+1}} \geq \frac{L_m L_{m+1}}{F_{n+2}};$$

$$\sum_{k=1}^m \frac{y_k^3}{L_m y_k + L_{m+1}y_{k+1}} \geq \frac{F_m F_{m+1}}{L_{m+2}}$$

for every positive integer n.

Solution by Ángel Plaza, Gran Canaria, Spain.

Both inequalities are consequence of the following Nesbitt type more general inequality where the left-hand side sum is cyclic

$$\sum_{k=1}^m \frac{x_k^3}{ax_k + bx_{k+1}} \geq \frac{\sum_{k=1}^m x_k^2}{a+b}.$$

Then the left-hand side of the fist equation, LHS is

$$\begin{aligned}LHS &\geq \frac{2}{F_{n+2}} + \frac{\sum_{k=1}^m x_k^2}{F_n + F_{n+1}}\\ &= \frac{2 + L_m L_{m+1} - 2}{F_{n+2}}\\ &= \frac{L_m L_{m+1}}{F_{n+2}},\end{aligned}$$

where we have used that $\sum_{k=1}^m L_k^2 = L_m L_{m+1} - 2$.

The second inequality is proved in the same way by now using that $\displaystyle\sum_{k=1}^m F_k^2 = F_m F_{m+1}$.

Also solved by Dmitry Fleischman and the proposers.

Identities With Sums of Ratios of Fibonacci Numbers and Products of Them

H-749 Proposed by Hideyuki Ohtsuka, Saitama, Japan.
(Vol. 52, No. 1, February 2014)

Let a, b, c and d be odd positive integers. If $a+b=c+d$, prove that

$$\sum_{k=1}^{b}\frac{F_a}{F_kF_{k+a}}+\sum_{k=1}^{a}\frac{F_b}{F_kF_{k+b}}=\sum_{k=1}^{d}\frac{F_c}{F_kF_{k+c}}+\sum_{k=1}^{c}\frac{F_d}{F_kF_{k+d}}.$$

Solution by Ángel Plaza, Gran Canaria, Spain.

We use the following reduction formula [1, Theorem 6]

$$\sum_{k=1}^{N}\frac{1}{F_kF_{k+a}}=\frac{1}{F_a}\sum_{k=1}^{\lfloor a/2\rfloor}\left(\frac{1}{F_{N+2k}F_{N+2k+1}}-\frac{1}{F_{2k}F_{2k+1}}\right)+\frac{\mathbb{K}_N}{F_a},$$

with $\mathbb{K}_N=\sum_{k=1}^{N}\frac{1}{F_kF_{k+1}}$. Therefore, the LHS and the RHS of the proposed identity are, respectively

$$\begin{aligned}LHS=&\sum_{k=1}^{\lfloor a/2\rfloor}\left(\frac{1}{F_{b+2k}F_{b+2k+1}}-\frac{1}{F_{2k}F_{2k+1}}\right)+\mathbb{K}_b\\&+\sum_{k=1}^{\lfloor b/2\rfloor}\left(\frac{1}{F_{a+2k}F_{a+2k+1}}-\frac{1}{F_{2k}F_{2k+1}}\right)+\mathbb{K}_a\end{aligned}$$

$$\begin{aligned}RHS=&\sum_{k=1}^{\lfloor c/2\rfloor}\left(\frac{1}{F_{d+2k}F_{d+2k+1}}-\frac{1}{F_{2k}F_{2k+1}}\right)+\mathbb{K}_d\\&+\sum_{k=1}^{\lfloor d/2\rfloor}\left(\frac{1}{F_{c+2k}F_{c+2k+1}}-\frac{1}{F_{2k}F_{2k+1}}\right)+\mathbb{K}_c.\end{aligned}$$

Since a, b, c and d are odd positive integers with $a+b=c+d$ we may assume that $a=2\alpha+1$, $c=a+2m$, $b=2\beta+1$ and $d=b-2m$. Then $\lfloor a/2\rfloor=\alpha$, $\lfloor b/2\rfloor=\beta$, $\lfloor c/2\rfloor=\alpha+m$, and $\lfloor d/2\rfloor=\beta-m$. Previous expressions for LHS and RHS are now

$$\begin{aligned}LHS=&\sum_{k=1}^{\alpha}\left(\frac{1}{F_{b+2k}F_{b+2k+1}}-\frac{1}{F_{2k}F_{2k+1}}\right)+\sum_{k=1}^{b}\frac{1}{F_kF_{k+1}}\\&+\sum_{k=1}^{\beta}\left(\frac{1}{F_{a+2k}F_{a+2k+1}}-\frac{1}{F_{2k}F_{2k+1}}\right)+\sum_{k=1}^{a}\frac{1}{F_kF_{k+1}}\end{aligned}$$

$$\begin{aligned}RHS=&\sum_{k=1}^{\alpha+m}\left(\frac{1}{F_{b-2m+2k}F_{b-2m+2k+1}}-\frac{1}{F_{2k}F_{2k+1}}\right)+\sum_{k=1}^{b-2m}\frac{1}{F_kF_{k+1}}\\&+\sum_{k=1}^{\beta-m}\left(\frac{1}{F_{a+2m+2k}F_{a+2m+2k+1}}-\frac{1}{F_{2k}F_{2k+1}}\right)+\sum_{k=1}^{a+2m}\frac{1}{F_kF_{k+1}},\end{aligned}$$

where after cancelling common terms we have to prove that $LHS^* = RHS^*$:

$$LHS^* = \sum_{k=b-2m+1}^{b} \frac{1}{F_k F_{k+1}} + \sum_{k=1}^{m} \frac{1}{F_{a+2k}F_{a+2k+1}}$$
$$RHS^* = \sum_{k=1}^{m} \frac{1}{F_{b-2m+2k}F_{b-2m+2k+1}} + \sum_{k=a+1}^{a+2m} \frac{1}{F_k F_{k+1}},$$

which are clearly the same.

References

[1] S. Rabinowitz, *Algorithmic summation of reciprocals of products of Fibonacci numbers*, The Fibonacci Quarterly, **37.2** (1999), 122–127.

Also solved by the proposer.

Identities With Generalized Tribonacci Recurrences

H-750 Proposed by Hideyuki Ohtsuka, Saitama, Japan. (Vol. 52, No. 1, February 2014)

Generalized Tribonacci sequences $\{R_n\}$ and $\{S_n\}$ are defined by

$$R_{n+3} = pR_{n+2} + qR_{n+1} + rR_n \qquad (\text{for } n \geq 0);$$
$$S_{n+3} = pS_{n+2} + qS_{n+1} + rS_n \qquad (\text{for } n \geq 0),$$

with arbitrary $p, q, r, R_0, R_1, R_2, S_0, S_1, S_2$. For positive integers a, b, c, d such that $a+b = c+d$, prove that

$$R_{a+3}S_{b+3}+qR_{a+2}S_{b+2}+prR_{a+1}S_{b+1}-r^2R_aS_b = R_{c+3}S_{d+3}+qR_{c+2}S_{d+2}+prR_{c+1}S_{d+1}-r^2R_cS_d.$$

Solution by the proposer.

We have

$$\begin{aligned}
&R_{a+3}S_{b+3} - R_{a+2}S_{b+4} + qR_{a+2}S_{b+2} - qR_{a+1}S_{b+3} \\
&= S_{b+3}(R_{a+3} - qR_{a+1}) - R_{a+2}(S_{b+4} - qS_{b+2}) \\
&= S_{b+3}(pR_{a+2} + rR_a) - R_{a+2}(pS_{b+3} + rS_{b+1}) \\
&= rS_{b+3}R_a - rR_{a+2}S_{b+1} \\
&= r(pS_{b+2} + qS_{b+1} + rS_b)R_a - r(pR_{a+1} + qR_a + rR_{a-1})S_{b+1} \\
&= r^2R_aS_b - r^2R_{a-1}S_{b+1} - prR_{a+1}S_{b+1} + prR_aS_{b+2}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&R_{a+3}S_{b+3} + qR_{a+2}S_{b+2} + prR_{a+1}S_{b+1} - r^2R_aS_b \\
&= R_{a+2}S_{b+4} + qR_{a+1}S_{b+3} + prR_aS_{b+2} - r^2R_{a-1}S_{b+1}.
\end{aligned}$$

Letting $A_{a,b} = R_{a+3}S_{b+3} + qR_{a+2}S_{b+1} + prS_{a+1}S_{b+1} - r^2R_aS_b$, we have $A_{a,b} = A_{a-1,b+1}$. Using this identity repeatedly,

$$\cdots = A_{a+2,b-2} = A_{a+1,b-1} = A_{a,b} = A_{a-1,b+1} = A_{a-2,b+2} = \cdots .$$

Thus, we have $A_{a,b} = A_{a-j,b+j}$. That is, $A_{a,b} = A_{c,d}$ for $a + b = c + d$. Therefore we obtain the desired identity.

Also solved by Dmitry Fleischman.

An Inequality With Sums of Binomial Coefficients and Fibonacci Numbers

H-751 Proposed by D. M. Bătineţu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania.
(Vol. 52, No. 2, May 2014)

Prove that

$$\left(\frac{F_{m+1}^{2n}+\binom{2n+1}{1}F_{m+1}^{2n-1}F_m+\cdots+\binom{2n+1}{n}F_{m+1}^{n}F_m^{n}}{F_m}\right)^{p+1} + \left(\frac{\binom{2n+1}{n+1}F_{m+1}^{n}F_m^{n}+\cdots+\binom{2n+1}{2n}F_{m+1}F_m^{2n-1}}{F_{m+1}}\right)^{p+1} \geq \frac{1}{2^p}\left(\frac{F_{m+2}^{2n+1}}{F_mF_{m+1}}\right)^{p+1}$$

holds for any $p \geq 0$ and positive integers m and n, and that the same inequality holds with all the F's replaced by L's.

Solution by Harris Kwong, SUNY, Fredonia.

The left-hand side of the inequality is in the form of $y_1^{p+1}+y_2^{p+1}$, where

$$y_1 = \frac{F_{m+1}^{2n}+\binom{2n+1}{1}F_{m+1}^{2n-1}F_m+\cdots+\binom{2n+1}{n}F_{m+1}^{n}F_m^{n}}{F_m}$$

$$[3pt]y_2 = \frac{\binom{2n+1}{n+1}F_{m+1}^{n}F_m^{n}+\cdots+\binom{2n+1}{2n}F_{m+1}F_m^{2n-1}+F_m^{2n}}{F_{m+1}}.$$

Notice that

$$F_mF_{m+1}(y_1+y_2) = (F_{m+1}+F_m)^{2n+1} = F_{m+2}^{2n+1}.$$

For any positive numbers $x_1, x_2, \ldots, x_k$, it is well-known that

$$f(r) = \left(\frac{x_1^r+x_2^r+\cdots+x_k^r}{k}\right)^{\frac{1}{r}}$$

is an increasing function of r. Hence,

$$\left(\frac{y_1^{p+1}+y_2^{p+1}}{2}\right)^{\frac{1}{p+1}} \geq \frac{y_1+y_2}{2} = \frac{1}{2}\left(\frac{F_{m+2}^{2n+1}}{F_mF_{m+1}}\right),$$

from which the desired inequality follows immediately, and it is clear that it also holds when all the F's are replaced by L's.

Also solved by Dmitry Fleischman, Kenneth B. Davenport, Zbigniew Jakubczyk, Ángel Plaza, and the proposers.

Errata: In the statement of Advanced Problem **H-751**, there was an additional term "$+F_m^{2n}$" in the numerator of the second fraction is the left–hand side of the inequality to be proven. The present solution takes this into account.