ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY
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PROBLEMS PROPOSED IN THIS ISSUE

H-813 Proposed by D. M. Bătinețu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania
If \( x_k > 0 \) for \( k = 1, \ldots, n \) and \( m \geq 0 \) is an integer, prove that
\[
\left( \sum_{k=1}^{n} \frac{1}{x_k} \right) \sum_{\text{cyclic}} \frac{x_1 x_2 x_3}{L_m x_2 x_3 + L_{m+1} x_3 x_1 + L_{m+2} x_1 x_2} \geq \frac{n^2}{2 L_{m+2}}
\]
and that the same inequality holds with the Lucas numbers replaced by the Fibonacci numbers.

H-814 Proposed by Ray Melham, Sydney, Australia
Define the Tribonacci numbers, for all integers \( n \), by \( T_n = T_{n-1} + T_{n-2} + T_{n-3} \), with \( T_{-1} = 0, T_0 = 0, \) and \( T_1 = 1 \). If \( k \) and \( n \) are integers, prove that
\[
-T_{2k} T_{n-2}^2 - T_{2k-2} T_{n-1}^2 - 2T_{2k-1} T_n^2 + 2(T_{2k} + T_{2k+1}) T_{n+1}^2 + (T_{2k} + 2T_{2k+1}) T_{n+2}^2 + T_{2k+2} T_{n+3}^2 = 2 T_{2n+2k+4}.
\]

H-815 Proposed by Mehtaab Sawhney, Commack, NY
Let \( p \) be a prime congruent to 1 modulo 4. Prove that
\[
\sum_{n=0}^{p-1} 2^n \binom{3n}{n} \equiv 0 \pmod{p}.
\]
Proposed by D. M. Bătinețu-Giurgiu, Bucharest and Neculai Stanciu, Buzău, Romania

Prove that for a positive integer \(n\)
\[
\frac{F_1}{(F_1^2 + F_2^2)^2} + \frac{F_2}{(F_1^2 + F_2^2 + F_3^2)^2} + \cdots + \frac{F_n}{(F_1^2 + F_2^2 + \cdots + F_{n+1}^2)^2} \geq \frac{1}{F_{n+2}} - \frac{1}{F_{n+2}^2}.
\]

SOLUTIONS

An identity with Fibonomial coefficients

Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 53, No. 4, November 2015)

Let \(\binom{n}{k}_F\) denote the Fibonomial coefficient. For integers \(n \geq 1\) and \(r \neq 0\) with \(n + r \neq 0\), prove that
\[
\sum_{k=0}^{n} (-1)^{k(k+1)/2} F_{k+r} \left(\frac{F_r}{F_{n+r}}\right)^k \binom{n}{k}_F = 0.
\]

Solution by the proposer

It is known that
\[
\sum_{k=0}^{n} (-1)^{k(k+1)/2} \binom{n}{k} x^k = \prod_{k=0}^{n-1} (1 - \alpha^{n-k-1} \beta^k x)
\]
(see [1]). Let \(c = \frac{F_r}{F_{n+r}}\). We have
\[
\sum_{k=0}^{n} (-1)^{k(k+1)/2} F_{k+r} e^k \binom{n}{k}_F = \sum_{k=0}^{n} (-1)^{k(k+1)/2} \frac{\alpha^r(c\alpha)^k - \beta^r(c\beta)^k}{\sqrt{5}} \binom{n}{k}_F \Rightarrow
\]
\[
= \frac{\alpha^r}{\sqrt{5}} \prod_{k=0}^{n-1} (1 - c\alpha^{n-k-1} \beta^k) - \frac{\beta^r}{\sqrt{5}} \prod_{k=0}^{n-1} (1 - \alpha^{n-k-1} \beta^{k+1}) \quad \text{(by (1))}
\]
\[
= \frac{\alpha^r}{\sqrt{5}} \prod_{k=0}^{n-1} (1 - c\alpha^{n-k-1} \beta^k) - \beta^r \sqrt{5} \prod_{k=1}^{n} (1 - \alpha^{n-k} \beta^k)
\]
\[
= \frac{1}{\sqrt{5}} \alpha^r (1 - c\alpha^n) - \beta^r (1 - c\beta^n) P(n),
\]
where \(P(1) = 1\) and \(P(n) = \prod_{k=1}^{n-1} (1 - \alpha^{n-k} \beta^k)\) for \(n \geq 2\).

Here, we have
\[
\alpha^r (1 - c\alpha^n) - \beta^r (1 - c\beta^n) = \alpha^r - c\alpha^{r+n} - \beta^r + c\beta^{r+n}
\]
\[
= \sqrt{5} (F_r - cF_{n+r}) = \sqrt{5} (F_r - F_r) = 0.
\]

Therefore, we obtain the desired identity.

Note: In the same manner, for integers \(n \geq 1\) and \(r\), we have
\[
\sum_{k=0}^{n} (-1)^{k(k+1)/2} L_{k+r} \left(\frac{L_r}{L_{n+r}}\right)^k \binom{n}{k}_F = 0.
\]

A closed form for a certain sum

**Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 53, No. 4, November 2015)**

Given real numbers $r$ and $t > 0$ and an integer $n \geq 0$, find a closed form expression for the sum:

$$\sum_{k=0}^{n} \frac{1}{f_k(L_{2k}^r + t)(L_{2k+1}^r + t) \cdots (L_{2n}^r + t)},$$

where $f_0 = t/(t + 1)$ and $f_k = F_{2k+1}^r$ for $k \geq 1$.

**Solution by the proposer**

We find the identity

$$\sum_{k=0}^{n} \frac{1}{f_k(L_{2k}^r + t)(L_{2k+1}^r + t) \cdots (L_{2n}^r + t)} = \frac{1}{tF_{2n+1}^r}.$$  \hspace{2cm} (2)

The proof of (2) is by mathematical induction on $n$. For $n = 0$, both sides are equal to $1/t$. Assume that (2) holds for $n$. For $n + 1$, we have

$$\sum_{k=0}^{n+1} \frac{1}{f_k(L_{2k}^r + t)(L_{2k+1}^r + t) \cdots (L_{2n+1}^r + t)} = \frac{1}{f_{n+1}(L_{2n+1}^r + t)} + \frac{1}{(L_{2n+1}^r + t)} \sum_{k=0}^{n} \frac{1}{f_k(L_{2k}^r + t)(L_{2k+1}^r + t) \cdots (L_{2n}^r + t)}$$

$$= \frac{1}{F_{2n+2}^r + tF_{2n+2}^r} + \frac{1}{(L_{2n+1}^r + t)} \times \frac{1}{tF_{2n+1}^r}$$

$$= \frac{F_{2n+1}^r(L_{2n+1}^r + t)}{tF_{2n+1}^r F_{2n+2}^r(L_{2n+1}^r + t)} = \frac{F_{2n+1}^r(L_{2n+1}^r + t)}{tF_{2n+1}^r F_{2n+2}^r} = \frac{1}{tF_{2n+2}^r}.$$

Thus, (2) holds for $n + 1$.

Also solved by Dmitry Fleischman.

**More closed form expressions**

**Proposed by Hideyuki Ohtsuka, Saitama, Japan (Vol. 53, No. 4, November 2015)**

Find a closed form expression for the sums:

(i) $\sum_{k=1}^{n} (L_{2k} \pm \sqrt{5})(L_{2k+1} \pm \sqrt{5}) \cdots (L_{2n} \pm \sqrt{5})$ for $n \geq 1$;

(ii) $\sum_{k=m+1}^{n} (L_{2k} \pm L_{2m})(L_{2k+1} \pm L_{2m}) \cdots (L_{2n} \pm L_{2m})$ for $n > m \geq 1$.

**Solution by the proposer**

We use the identity

$$L_m^2 = L_{2m} + 2(-1)^m \quad \text{(see [1](17c)).}$$  \hspace{2cm} (3)

For $n \geq 1$, we have

$$x^2 + x - 2 + (L_{2n} - x)(L_{2n} + x) = L_{2n}^2 + x - 2 = L_{2n+1} + x \quad \text{(by (3)).}$$
If $a_n = L_{2n} - x$, $b_n = L_{2n} + x$, and $c = x^2 + x - 2$, then we have $b_{n+1} = c + a_nb_n$. Using this identity repeatedly for $n \geq m + 2 \geq 2$, we have

$$b_{n+1} = c + a_nb_n = c + a_n(c + a_{n-1}b_{n-1}) = \cdots$$

$$= c + a_n(c + a_{n-1}(c + a_{n-2}(c + \cdots a_m + b_m + b_{m+1} \cdots)))$$

$$= c + \sum_{k=m+2}^{n} c \prod_{j=k}^{n} a_j + b_{m+1} \prod_{j=m+1}^{n} a_j.$$

Therefore, we obtain

$$(x^2 + x - 2) \sum_{k=m+2}^{n} \prod_{j=k}^{n} (L_{2j} - x) + (L_{2m+1} + x) \prod_{j=m+1}^{n} (L_{2j} - x) = L_{2n+1} - x^2 + 2. \quad (4)$$

(i) If $m = 0$ and $x = \pm \sqrt{5}$ in (4), for $n \geq 2$, we have

$$(3 \mp \sqrt{5}) \sum_{k=2}^{n} \prod_{j=k}^{n} (L_{2j} \pm \sqrt{5}) + (3 \mp \sqrt{5}) \prod_{j=1}^{n} (L_{2j} \pm \sqrt{5}) = L_{2n+1} - 3.$$

Therefore, we obtain

$$\sum_{k=1}^{n} \prod_{j=k}^{n} (L_{2j} \pm \sqrt{5}) = \frac{L_{2n+1} - 3}{3 \mp \sqrt{5}}.$$ 

This identity holds also for $n = 1$, since then,

$$RHS = \frac{L_4 - 3}{3 \mp \sqrt{5}} = 3 \pm \sqrt{5} = L_2 \pm \sqrt{5} = LHS.$$

(ii) If $m \geq 1$ and $x = \mp L_{2m}$ in (4), for $n \geq m + 2$, we have

$$(L_{2m}^2 \mp L_{2m} - 2) \sum_{k=m+2}^{n} \prod_{j=k}^{n} (L_{2j} \pm L_{2m}) + (L_{2m+1} \mp L_{2m}) \prod_{j=m+1}^{n} (L_{2j} \pm L_{2m})$$

$$= L_{2n+1} - L_{2m}^2 + 2.$$ 

Using (3), we have

$$(L_{2m+1} \mp L_{2m}) \sum_{k=m+1}^{n} \prod_{j=k}^{n} (L_{2j} \pm L_{2m}) = L_{2n+1} - L_{2m+1}.$$ 

Therefore, we obtain

$$\sum_{k=m+1}^{n} \prod_{j=k}^{n} (L_{2j} \pm L_{2m}) = \frac{L_{2n+1} - L_{2m+1}}{L_{2m+1} \mp L_{2m}}.$$ 

The identity holds for $n = m + 1$ as well, since then,

$$RHS = \frac{L_{2m+2} - L_{2m+1}}{L_{2m+1} \mp L_{2m}} = \frac{L_{2m+1} - L_{2m}^2}{L_{2m+1} \mp L_{2m+1}} = L_{2m+1} \pm L_{2m} = RHS,$$

where in the above chain of equalities we used (3).


**And yet more closed form formulas**
Using the above identity, we have

\[ \sum_{n=1}^{\infty} \alpha(s-1)^n F_{rn} F_{r(n+1)} F_{r(n+2)} \cdots F_{r(n+s)} \]

and

\[ \sum_{n=1}^{\infty} \alpha(s-1)^n L_{rn} L_{r(n+1)} L_{r(n+2)} \cdots L_{r(n+s)} \]

**Solution by the proposer**

(i) We have

\[
\frac{\beta^{srn}}{F_{rn} F_{r(n+1)} \cdots F_{r(n+s-1)}} = \frac{\beta^{sr(n+1)}}{F_{r(n+1)} F_{r(n+2)} \cdots F_{r(n+s)}} = \frac{\beta^{srn} (F_{r(n+s)} - \beta^{sr} F_{rn})}{\sqrt{5} F_{rn} F_{r(n+1)} \cdots F_{r(n+s)}} \]

\[ = \frac{(-1)^{srn} \alpha^{rn} (\alpha^{sr} - \beta^{sr})}{\sqrt{5} \alpha^{rn} F_{sr} F_{r(n+1)} \cdots F_{r(n+s)}}. \]

Using the above identity, we have

\[
\sum_{n=1}^{m} \frac{(-1)^{srn} \alpha^{rn}}{\prod_{i=n}^{n+s} F_{ri}} = \frac{1}{F_{sr}} \sum_{n=1}^{m} \left( \frac{\beta^{srn}}{\prod_{i=n}^{n+s-1} F_{ri}} - \frac{\beta^{sr(n+1)}}{\prod_{i=n+1}^{n+s} F_{ri}} \right) \]

\[ = \frac{1}{F_{sr}} \left( \frac{\beta^{sr}}{\prod_{i=1}^{n} F_{ri}} - \frac{\beta^{sr(m+1)}}{\prod_{i=m+1}^{n+s} F_{ri}} \right). \]

Therefore, we obtain

\[ \sum_{n=1}^{\infty} \frac{(-1)^{srn} \alpha^{rn}}{\prod_{i=n}^{n+s} F_{ri}} = \frac{\beta^{sr}}{F_{sr} F_{2r} F_{3r} \cdots F_{sr}}. \]

(ii) We have

\[
\frac{\beta^{srn}}{L_{rn} L_{r(n+1)} \cdots L_{r(n+s-1)}} = \frac{\beta^{sr(n+1)}}{L_{r(n+1)} L_{r(n+2)} \cdots L_{r(n+s)}} = \frac{\beta^{srn} (L_{r(n+s)} - \beta^{sr} L_{rn})}{\sqrt{5} F_{rn} L_{r(n+1)} \cdots L_{r(n+s)}} \]

\[ = \frac{(-1)^{srn} \alpha^{rn} (\alpha^{sr} - \beta^{sr})}{\sqrt{5} \alpha^{rn} L_{sr} L_{r(n+1)} \cdots L_{r(n+s)}}. \]

Using the above identity, we have

\[
\sum_{n=1}^{m} \frac{(-1)^{srn} \alpha^{rn}}{\prod_{i=n}^{n+s} L_{ri}} = \frac{1}{\sqrt{5} F_{sr}} \sum_{n=1}^{m} \left( \frac{\beta^{srn}}{\prod_{i=n}^{n+s-1} L_{ri}} - \frac{\beta^{sr(n+1)}}{\prod_{i=n+1}^{n+s} L_{ri}} \right) \]

\[ = \frac{1}{\sqrt{5} F_{sr}} \left( \frac{\beta^{sr}}{\prod_{i=1}^{n} L_{ri}} - \frac{\beta^{sr(m+1)}}{\prod_{i=m+1}^{n+s} L_{ri}} \right). \]
Therefore, we obtain
\[
\sum_{n=1}^{\infty} \frac{(-1)^{sn} \alpha^{(s-1)rn} L_{rn} L_{r(n+1)} L_{r(n+2)} \cdots L_{r(n+s)}}{\sqrt{5} F_{sr} (L_r L_{2r} L_{3r} \cdots L_{sr})} = \frac{\beta^{sr}}{L_r L_{n+1} L_{n+2} L_{n+3} L_{n+4}}.
\]

**Example.** If \( s = 4 \) and \( r = 1 \), then we have
\[
\sum_{n=1}^{\infty} \frac{\alpha^{3n} F_{n} F_{n+1} F_{n+2} F_{n+3} F_{n+4}}{\sqrt{5} F_{sr} (L_r L_{2r} L_{3r} \cdots L_{sr})} = \frac{7 - 3\sqrt{5}}{36};
\]
\[
\sum_{n=1}^{\infty} \frac{\alpha^{3n} L_{n} L_{n+1} L_{n+2} L_{n+3} L_{n+4}}{\sqrt{5} F_{sr} (L_r L_{2r} L_{3r} \cdots L_{sr})} = \frac{-15 + 7\sqrt{5}}{2520}.
\]

Also solved by Dmitry Fleischman.