ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, IMATE, UNAM, AP. POSTAL 61-3 (XANGARI), CP 58 089, MORELIA, MICHOACAN, MEXICO, or by e-mail at fluca@matmor.unam.mx as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-618 Proposed by Slavko Simic, Mathematical Institute SANU, Belgrade

Prove that there exists a constant $c \geq 2.5$ such that the inequality

$$e^x \geq 1 + x^\alpha$$

holds for each $x \geq 0$ if and only if $\alpha \in [1, c]$. What is the value of c?

H-619 Proposed by Jayantibhai M. Patel, Ahmedabad, India

For any positive integer $n \geq 2$, prove that the value of the following determinant

$$\begin{vmatrix} -(6F_{n-1}^2 - L_n^2) & 2F_{n+1}F_{n+2} & 2F_nF_{n+2} & 2F_{n-1}F_{n+2} & 2F_{n-2}F_{n+2} \\ 2F_{n+2}F_{n+1} & -(2F_{n-1}^2 + 5F_n^2) & 2F_nF_{n+1} & 2F_{n-1}F_{n+1} & 2F_{n-2}F_{n+1} \\ 2F_{n+2}F_n & 2F_{n+1}F_n & -(4F_n^2 + L_n^2) & 2F_{n-1}F_n & 2F_{n-2}F_n \\ 2F_{n+2}F_{n-1} & 2F_{n+1}F_{n-1} & 2F_nF_{n-1} & -(5F_n^2 + 2F_{n+1}^2) & 2F_{n-2}F_{n-1} \\ 2F_{n+2}F_{n-2} & 2F_{n+1}F_{n-2} & 2F_nF_{n-2} & 2F_{n-1}F_{n+2} & -(6F_{n+1}^2 - F_n^2) \end{vmatrix}$$

is $(6F_n^2 + L_n^2)^5$.

H-620 Proposed by José Luis Díaz-Barrero, Barcelona, Spain and Óscar Ciaurri Ramírez, Logroño, Spain

Let ABC be a triangle. Prove that the following inequality holds for $\alpha \in [0, \pi/2)$:

$$\sqrt{F_{n+1}F_{n+2}}\cos(C-\alpha) + \sqrt{F_{n+2}F_n}\cos(B-\alpha) + \sqrt{F_nF_{n+1}}\cos(A-\alpha) \leq 2F_{n+2}\cos\Big(\frac{\pi}{3} - \alpha\Big).$$

H-621 Proposed by Mario Catalani, Torino, Italy

Let $L_n(x,y)$ be the bivariate Lucas polynomials, defined by $L_n(x,y) = xL_{n-1}(x,y) + yL_{n-2}(x,y)$, $L_0(x,y) = 2$, $L_1(x,y) = x$. Assume $x^2 + 4y \neq 0$. Prove the following identity

$$\sum_{k=0}^{n} \binom{n+k}{k} (-y)^k x^{-(k+1)} L_{n+1-k}(x,y) = x^n.$$

SOLUTIONS

A product of Pell and Fibonacci

H-605 Proposed by José Luis Díaz-Barrero and Juan José Egozcue, Spain (Vol. 41, no. 5, November 2003)

Find the smallest integer k for which $\lambda_0 a_n + \lambda_1 a_{n+1} + \cdots + \lambda_k a_{n+k} = 0$ holds for all $n \geq 1$ with some integers $\lambda_0, \ldots, \lambda_k$ not all zero, where $\{a_n\}_{n\geq 1}$ is the integer sequence defined by

$$a_n = \Big(\sum_{\ell=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2\ell+1} 2^\ell \Big) \Big(\sum_{\ell=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{2^{n-1}} \binom{n}{2\ell+1} 5^\ell \Big).$$

Solution by H.-J. Seiffert, Berlin, Germany

It is well-known that

$$\sum_{\ell=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2\ell+1} 2^\ell = P_n \qquad \text{and} \qquad \sum_{\ell=0}^{\lfloor (n-1)/2 \rfloor} \frac{1}{2^{n-1}} \binom{n}{2\ell+1} 5^\ell = F_n,$$

where $\{P_n\}_{n\geq 0}$ is the sequence of Pell numbers defined recursively by $P_{n+2} = 2P_{n+1} + P_n$ for $n \geq 0$, $P_0 = 0$, and $P_1 = 1$. Thus, we have $a_n = P_n F_n$ for all $n \geq 1$. From [1], we know that

$$a_n - 2a_{n+1} - 7a_{n+2} - 2a_{n+3} + a_{n+4} = 0 \qquad \text{for all } n \geq 1. \tag{1}$$

Using the values $a_1 = 1,\ a_2 = 2,\ a_3 = 10,\ a_4 = 36,\ a_5 = 145,\ a_6 = 560$ and $a_7 = 2197$, it is easily checked that for $k = 0,\ 1,\ 2,$ and 3, respectively, the determinant of the matrix $(a_{i+j-1})_{i,j=1,\ldots,k+1}$ has the values 1, 6, 14 and 9, respectively, so that, by Cramer's rule, the only solution of the system of linear equations

$$\lambda_0 a_n + \lambda_1 a_{n+1} + \cdots + \lambda_k a_{n+k} = 0, \qquad \text{for } n = 1, \ldots, k+1,$$

is $\lambda_0 = \lambda_1 = \cdots = \lambda_k = 0$ for such k. Hence, by (1), $k = 4$ is the smallest nonnegative integer asked for.

[1] "Problem B-625", The Fibonacci Quarterly **27.4** (1989): 376.

Also solved by V. Mathe the proposers.

Another binomial identity

H-606 Proposed by Mario Catalani, University of Torino, Italy (Vol. 42, no. 1, February 2004)

Let us consider, for a nonnegative integer n, the following sum

$$S_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{2\lfloor \frac{k}{2} \rfloor} - \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor - 1} \binom{n-1-k}{2\lfloor \frac{k}{2} \rfloor + 1}.$$

A summation with a negative upper limit is taken to be equal to zero. Express S_n both in closed form and as a recurrence.

Solution by H.-J. Seiffert, Berlin, Germany

It is known (see [1]) that, for all nonnegative integers n,

$$T_n = \sum_{r=0}^{\lfloor (n-1)/2 \rfloor} (-1)^r \binom{n-1-r}{r} = \frac{2}{\sqrt{3}} \sin\Big(\frac{n\pi}{3}\Big).$$

Considering the cases in which the index k is even and odd, we then have

$$S_n = \sum_{0 \le 2j \le \lfloor n/2 \rfloor} \binom{n-2j}{2j} + \sum_{0 \le 2j+1 \le \lfloor n/2 \rfloor} \binom{n-2j-1}{2j}$$

$$- \sum_{0 \le 2j \le \lfloor n/2 \rfloor - 1} \binom{n-1-2j}{2j+1} - \sum_{0 \le 2j+1 \le \lfloor n/2 \rfloor - 1} \binom{n-2j-2}{2j+1}$$

$$= \sum_{0 \le r \le \lfloor n/2 \rfloor} (-1)^r \binom{n-r}{r} + \sum_{0 \le r \le \lfloor n/2 \rfloor - 1} (-1)^r \binom{n-1-r}{r}$$

$$= T_{n+1} + T_n - \frac{1}{2}(-1)^{\lfloor n/2 \rfloor}(1-(-1)^n).$$

¿From known trigonometric relations, one finds

$$U_n = T_{n+1} + T_n = 2\sin\Big(\frac{(2n+1)\pi}{6}\Big),$$

so that the closed form expression

$$S_n = 2\sin\Big(\frac{(2n+1)\pi}{6}\Big) - \frac{1}{2}(-1)^{\lfloor n/2\rfloor}(1+(-1)^n)$$

holds. Again, by known trigonomeric relations, $U_{n+2} = U_{n+1} - U_n$. Now, it is easily verified that the recurrence

$$S_{n+2} = S_{n+1} - S_n + \frac{1}{2}(-1)^{\lfloor (n+1)/2\rfloor}(1+(-1)^n)$$

holds.

Editor's comment. Each solver submitted a different looking, yet mathematically equivalent, closed form for S_n. For example,

$$S_{2n+1} = (-1)^{\lfloor (2n+1)/3\rfloor} - (-1)^{\lfloor (2n+1)/2\rfloor} + \frac{1}{2}\Big((-1)^{\lfloor 2n/3\rfloor} + (-1)^{\lfloor (2n+2)/3\rfloor}\Big) \qquad \text{(proposer)},$$

$$S_n = -\sin\Big(\frac{n\pi}{2}\Big) + 2\cos\Big(\frac{(n-1)\pi}{3}\Big) \qquad \text{(Bruckman)}.$$

Also, the proposer proved the recurrence

$$S_{2n+1} = -2S_{2(n-1)+1} - 2S_{2(n-2)+1} - S_{2(n-3)+1} \qquad \text{for all } n \geq 3,$$

and a similar type of recurrence for $(S_{2n})_{n\geq 3}$.

[1] "Solution to Problem B-828", The Fibonacci Quarterly **36.1** (1998): 87–88.

Also solved by Paul Bruckman and the proposer.

Dividing differences

H-607 Proposed by José Luis Díaz-Barrero, Barcelona, Spain

Let n be a positive integer greater than or equal to 3. Evaluate the sum

$$\sum_{i=1}^{n}\Big[\Big(\frac{F_{i+1}-F_{i-1}}{F_{i+2}^2-F_{i-2}^2}\Big)^{n-2}\prod_{\substack{j=1\\ j\neq i}}\Big(1-\frac{F_{j+2}-F_{j-2}}{F_{i+2}-F_{i-2}}\Big)^{-1}\Big].$$

Solution by Paul Bruckman, Sointula, Canada

We employ the following identities

$$F_{i+1} - F_{i-1} = F_i, \qquad F_{i+2} - F_{i-2} = L_i \qquad \text{and} \qquad F_{i+2} + F_{i-2} = 3F_i.$$

Denote the given expression by S_n. Then,

$$S_n = \sum_{i=1}^{n} \Big(\frac{F_i}{3F_iL_i}\Big)^{n-2} \prod_{\substack{j=1\\ j\neq i}}^{n} \Big(1 - \frac{L_j}{L_i}\Big)^{-1}.$$

After simplification, we get

$$S_n = \frac{1}{3^{n-2}} \sum_{i=1}^{n} L_i \prod_{\substack{j=1\\ j\neq i}}^{n} \Big(\frac{1}{L_i - L_j}\Big).$$

Thus,

$$S_n = \frac{1}{3^{n-2}} \Delta^{(n-1)}(z)[L_1, \ldots, L_n],$$

where $\Delta^{(n-1)}(z)[L_1, \ldots, L_n]$ is the divided difference of order $n-1$ of the polynomial function z evaluated at the points $L_1, \ldots, L_n$. Since $n \geq 3$, it follows that $S_n = 0$.

Also solved by Ovidiu Furdui, H.-J. Seiffert and the proposer.

Pell does it again

H-608 Proposed by Mario Catalani, University of Torino, Italy (Vol. 42, no. 1, February 2004)

Let P_n denote the Pell numbers

$$P_n = 2P_{n-1} + P_{n-2}, \;\; P_0 = 0, \;\; P_1 = 1.$$

Find

$$\lim_{n\to\infty} \prod_{k=1}^{n} \Big(1 + \frac{1}{\sqrt{2P_{2^k}^2 + 1}}\Big).$$

Solution by Kenneth B. Davenport, Frackville, PA, USA

Since

$$P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}}, \qquad \text{where } \alpha = 1+\sqrt{2},\ \beta = 1-\sqrt{2},$$

the given product may be expressed as

$$\prod_{k=1}^{n}\left(1+\frac{1}{\sqrt{\frac{2(\alpha^{2^k}-\beta^{2^k})^2}{(2\sqrt{2})^2}+1}}\right) = \prod_{k=1}^{n}\left(1+\frac{1}{\sqrt{\frac{(\alpha^{2^k}-\beta^{2^k})^2+4}{4}}}\right)$$

$$= \prod_{k=1}^{n}\left(1+\frac{2}{\sqrt{\alpha^{2^{k+1}}+\beta^{2^{k+1}}+2}}\right) = \prod_{k=1}^{n}\left(1+\frac{2}{\alpha^{2^k}+\beta^{2^k}}\right).$$

Since $\alpha\beta = -1$, this last product can be rewritten as

$$\left(1+\frac{2\beta^2}{\beta^4+1}\right)\left(1+\frac{2\beta^4}{\beta^8+1}\right)\cdots\left(1+\frac{2\beta^{2^n}}{\beta^{2^{n+1}}+1}\right).$$

The product can now be expressed as

$$\frac{(\beta^2+1)^2}{(\beta^4+1)}\cdot\frac{(\beta^4+1)^2}{(\beta^8+1)}\cdots\frac{(\beta^{2^n}+1)^2}{(\beta^{2^{n+1}}+1)}$$

$$= (\beta^2+1)^2(\beta^4+1)\ldots(\beta^{2^n}+1)(\beta^{2^{n+1}}+1)^{-1}$$

$$= \frac{(\beta^2+1)(1-\beta^{2^{n+1}})}{(1-\beta^2)(1+\beta^{2^{n+1}})}.$$

Since $|\beta| < 1$, we deduce easily that the desired product converges to

$$\frac{1+\beta^2}{1-\beta^2} = \sqrt{2}.$$

Editor's Comment. Both W. Janous and H.-J. Seiffert proved appropriate generalizations of this problem when the sequence $2P_{2^k}^2 + 1$ is replaced by a quadratic expression in $(x_{2^k})_{k\geq 1}$, where $(x_k)_{k\geq 0}$ is a nondegenerate binary recurrent sequence with $x_0 = 0$ whose characteristic roots are real quadratic units.

Also solved by Paul Bruckman, Ovidiu Furdui, W. Janous, H.-J. Seiffert and the proposer.