ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Florian Luca

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, IMATE, UNAM, AP. POSTAL 61-3 (XANGARI), CP 58 089, MORELIA, MICHOACAN, MEXICO, or by e-mail at fluca@matmor.unam.mx as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

Dedicated to RAYMOND E. WHITNEY

H-622 Proposed by Lawrence Somer, The Catholic University of America, Washington, DC

Let $u(a,b)$ be the Lucas sequence defined by $u_0 = 0,\ u_1 = 1$ and $u_{n+2} = au_{n+1} - bu_n$ and having discriminant $D(a,b) = a^2 - 4b$, where a and b are integers. Let $\omega(n)$ denote the number of distinct prime divisors of n. Let $c > 1$ be a fixed positive integer. Show that there exist $2^{\omega(c)}$ distinct Lucas sequences $u(a,c^2)$ such that $2p+1 \mid u_p(a,c^2)$ for every Sophie Germain prime p such that $2p+1 \nmid D(a,c^2)$, where $(a,c) = 1$ and $aD(a,c^2) \neq 0$. Recall that p is a Sophie Germain prime if both p and $2p+1$ are primes.

H-623 Proposed by José Luis Díaz-Barrero, Barcelona, Spain

Let n be a positive integer. Prove that

$$\prod_{k=1}^{2n-1} (2n-k)^{F_k^2} \le \Big(\frac{F_{2n}}{F_{2n-1}}\Big)^{F_{2n-1}F_{2n}}.$$

H-624 Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI

Prove that

$$\lim_{n\to\infty}\int_0^1 \frac{1+t^{F_n}}{1+t^{L_n}}dt = 1 \quad \text{and} \quad \lim_{n\to\infty}\int_1^n \frac{1+t^{F_n}}{1+t^{L_n}}dt = 0.$$

H-625 Proposed by Russel Jay Hendel, Townson University, MD

For an integer $m > 0$ let K_m be the smallest positive integer such that $F_{n+m} < K_m F_n$ holds for all large n. For example, $K_1 = 2$ because $F_n < F_{n+1} < 2F_n$ holds for all large n. Provide an explicit formula for K_m.

H-626 Proposed by H.-J. Seiffert, Berlin, Germany

The Fibonacci polynomials are defined by $F_0(x) = 0$, $F_1(x) = 1$, and $F_{n+2}(x) = xF_{n+1}(x) + F_n(x)$ for $n \geq 0$. Let n be a positive integer.

a. Prove that, for all complex numbers x,

$$F_{n+1}(x) + iF_n(x) = 4^{-n} \sum_{k=0}^{n} \binom{2n+1}{2k+1} (x-2i)^k (x+2i)^{n-k}, \quad \text{where } i = \sqrt{-1}.$$

b. Deduce the identites

$$P_n = 2^{-\lfloor \frac{n}{2} \rfloor} \sum_{\substack{k=0 \\ 4 \nmid n-2k+1}}^{n-1} (-1)^{\lfloor \frac{(n-2k)}{4} \rfloor} \binom{2n-1}{2k+1} \text{ and } P_n = 2^{-\lfloor \frac{(n+1)}{2} \rfloor} \sum_{\substack{k=0 \\ 4 \nmid n-2k}}^{n} (-1)^{\lfloor \frac{(n-2k-1)}{4} \rfloor} \binom{2n+1}{2k+1},$$

where $P_n = F_n(2)$ is the nth Pell number.

SOLUTIONS

Fibonacci vectors

H-610 Proposed by Jayantibhai M. Patel, Ahmedabad, India (Vol. 42, no. 2, May 2004)

If $\overline{\boldsymbol{x}} = (-2F_n^2, 2F_{2n}, L_n^2)$, $\overline{\boldsymbol{y}} = (2F_{2n}, 2F_n^2 - L_n^2, 2F_{2n})$ and $\overline{\boldsymbol{z}} = (L_n^2, 2F_{2n}, -2F_n^2)$ are three vectors, then prove that

a. $\overline{\boldsymbol{x}}$, $\overline{\boldsymbol{y}}$, $\overline{\boldsymbol{z}}$ are mutually perpendicular vectors.

b. $||\overline{\boldsymbol{x}}|| = ||\overline{\boldsymbol{y}}|| = ||\overline{\boldsymbol{z}}|| = 2F_n^2 + L_n^2$.

c. $\overline{\boldsymbol{x}} \cdot (\overline{\boldsymbol{y}} \times \overline{\boldsymbol{z}}) = ||\overline{\boldsymbol{x}}||^3 = (2F_n^2 + L_n^2)^3$.

Based on solutions by Charles K. Cook, University of South Carolina Sumter, Sumter, SC, and Ovidiu Furdui, Western Michigan University, Kalamazoo, MI

The wellknown formula $F_{2n} = F_n L_n$ will be used as needed.

a.

$$\overline{\boldsymbol{x}} \cdot \overline{\boldsymbol{y}} = -4F_n^2 F_{2n} + 4F_{2n} F_n^2 - 2F_{2n} L_n^2 + 2L_n^2 F_{2n} = 0.$$

$$\overline{\boldsymbol{x}} \cdot \overline{\boldsymbol{z}} = -2F_n^2 L_n^2 + 4F_{2n}^2 - 2L_n^2 F_n^2 = -4F_n^2 L_n^2 + 4F_n^2 L_n^2 = 0.$$

$$\overline{\boldsymbol{y}} \cdot \overline{\boldsymbol{z}} = 2F_{2n} L_n^2 + 4F_n^2 F_{2n} - 2F_{2n} L_n^2 - 4F_n^2 F_{2n} = 0.$$

Thus, the three vectors are mutually perpendicular.

b.

$$||\overline{\boldsymbol{x}}||^2 = 4F_n^4 + 4F_{2n}^2 + L_n^4 = 4F_n^4 + 4F_n^2L_n^2 + L_n^4 = (2F_n^2 + L_n^2)^2.$$

$$||\overline{\boldsymbol{y}}||^2 = 4F_{2n}^2 + 4F_n^4 - 4F_n^2L_n^2 + L_n^4 + 4F_{2n}^2 = 4F_n^4 + 4F_n^2L_n^2 + L_n^4 = (2F_n^2 + L_n^2)^2.$$

$$||\overline{\boldsymbol{z}}||^2 = L_n^4 + 4F_{2n}^2 + 4F_n^4 = L_n^4 + 4F_n^2L_n^2 + L_n^4 = (2F_n^2 + L_n^2)^2.$$

Taking square roots, yields the desired results.

c. The fact that $||\overline{\boldsymbol{x}}||^3 = (2F_n^2 + L_n^2)^3$ follows from **b.** Since $\overline{\boldsymbol{x}}$, $\overline{\boldsymbol{y}}$ and $\overline{\boldsymbol{z}}$ are all mutually orthogonal and have equal norms, it follows that $\overline{\boldsymbol{y}} \times \overline{\boldsymbol{z}}$ equals $\pm||\overline{\boldsymbol{x}}||\overline{\boldsymbol{x}}$ (see [1]). We thus get that $\overline{\boldsymbol{x}} \cdot (\overline{\boldsymbol{y}} \times \overline{\boldsymbol{z}}) = \pm(2F_n^2 + L_n^2)^3$. It remains to establish that the sign is plus. For this, writing $\overline{\boldsymbol{i}} = (1,0,0)$, $\overline{\boldsymbol{j}} = (0,1,0)$ and $\overline{\boldsymbol{k}} = (0,0,1)$, we get

$$\overline{\boldsymbol{y}} \times \overline{\boldsymbol{z}} = \begin{vmatrix} \overline{\boldsymbol{i}} & \overline{\boldsymbol{j}} & \overline{\boldsymbol{k}} \\ 2F_{2n} & 2F_n^2 - L_n^2 & 2F_{2n} \\ L_n^2 & 2F_{2n} & -2F_n^2 \end{vmatrix}$$

$$= (-4F_n^4 + 2F_n^2L_n^2 - 4F_{2n}^2, 2F_{2n}L_n^2 + 4F_{2n}F_n^2, 4F_{2n}^2 - 2F_n^2L_n^2 + L_n^4)$$
$$= (-4F_n^4 - 2F_n^2L_n^2, 2F_{2n}(L_n^2 + 2F_n^2), 2F_n^2L_n^2 + L_n^4) = (2F_n^2 + L_n^2)\overline{\boldsymbol{x}},$$

which completes the proof of **c.**

[1] Manuel Schwartz, Simon Green and W.A. Ruthledge "Vector Analysis", Harper & Brothers, New York, 1960.

Also solved by Paul Bruckman, Kenneth Davenport and Walther Janous.

<u>A Fibonacci Series</u>

<u>H-611</u> Proposed by Ó. Ciaurri Ramírez, Logroño, Spain and J.L. Díaz-Barrero, Barcelona, Spain

(Vol. 42, no. 2, May 2004)

Evaluate

$$\sum_{n=0}^{\infty} \frac{1}{(2\alpha)^n(n+2)} \sum_{k=0}^{n} \frac{F_{k+1}F_{n-k+1}}{k+1},$$

where α denotes the golden section.

Solution by the proposers

First, we observe that if we set

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n+2} \sum_{k=0}^{n} \frac{F_{k+1}F_{n-k+1}}{k+1},$$

then the given sum will be obtained evaluating $f(x)$ at $x = 1/2\alpha$. Integrating the wellknown the identity

$$\sum_{n=0}^{\infty} F_{n+1}x^n = \frac{1}{1-x-x^2},$$

which is valid for $|x| < \frac{1}{\alpha}$, yields

$$\frac{1}{\sqrt{5}}\log\Big(\frac{1+x\alpha^{-1}}{1-x\alpha}\Big) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} F_{n+1}.$$

Now, using the Cauchy product of two series, we obtain

$$\frac{1}{\sqrt{5}}\log\Big(\frac{1+x\alpha^{-1}}{1-x\alpha}\Big)\frac{1}{1-x-x^2} = \sum_{n=0}^{\infty} x^{n+1} \sum_{k=0}^{n} \frac{F_{k+1}F_{n-k+1}}{k+1}.$$

Finally, integrating both sides of the preceding equality, we get

$$\frac{1}{10}\Big[\log\Big(\frac{1+x\alpha^{-1}}{1-x\alpha}\Big)\Big]^2 = \sum_{n=0}^{\infty} \frac{x^{n+2}}{n+2} \sum_{k=0}^{n} \frac{F_{k+1}F_{n-k+1}}{k+1} = x^2 f(x).$$

In this way, setting $x = \frac{1}{2\alpha}$ into the above expression, we get

$$\sum_{n=0}^{\infty} \frac{1}{(2\alpha)^n(n+2)} \sum_{k=0}^{n} \frac{F_{k+1}F_{n-k+1}}{k+1} = f\Big(\frac{1}{2\alpha}\Big) = \frac{2\alpha^2}{5}\Big[\log\Big(\frac{1+2\alpha^2}{\alpha^2}\Big)\Big]^2.$$

Also solved by Paul Bruckman and Walther Janous.

A misterious determinant

H-612 Proposed by Mario Catalani, University of Torino, Italy (Vol. 42, no. 3, August 2004)

Let a_r be the sequence $a_r = a_{r-1} + 2r$ for $r \geq 1$, with $a_0 = 0$. Let $\boldsymbol{A}_n$ be the matrix elements $a_{ij} = \min(i, j)$, $1 \leq i, j \leq n$, and let $\boldsymbol{I}$ be the identity matrix. Find

$$b_n = |\boldsymbol{A}_n + a_r\boldsymbol{I}|$$

as a function of r and n, where $|\cdot|$ is the determinant operator.

Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY

It follows from its recurrence that $a_r = 2r + 2(r-1) + \cdots + 2 = r(r+1)$. Observe that $\boldsymbol{A}_n^{-1}$ is a tridiagonal matrix:

$$\boldsymbol{A}_n^{-1} = \begin{bmatrix} 2 & -1 & 0 & & \\ -1 & 2 & -1 & & \\ 0 & -1 & 2 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & -1 & 2 & -1 \\ & & 0 & -1 & 1 \end{bmatrix}.$$

Adding the nth column of $\boldsymbol{A}_n^{-1}$ to its $(n-1)$th column, and then expanding the determinant using the last row, we find $|\boldsymbol{A}_n^{-1}| = |\boldsymbol{A}_{n-1}^{-1}| = \ldots = |\boldsymbol{A}_1^{-1}| = 1$. Thus, $b_n = |\boldsymbol{B}_n|$, where $\boldsymbol{B}_n = \boldsymbol{A}_n^{-1}(\boldsymbol{A}_n + a_r\boldsymbol{I}) = \boldsymbol{I} + a_r\boldsymbol{A}_n^{-1}$. Note that

$$\boldsymbol{B}_n = \begin{bmatrix} 2a_r+1 & -a_r & 0 & & \\ -a_r & 2a_r+1 & -a_r & & \\ 0 & -a_r & 2a_r+1 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & -a_r & 2a_r+1 & -a_r \\ & & 0 & -a_r & a_r+1 \end{bmatrix}.$$

Its cofactor expansion along the first row yields the recurrence relation

$$b_n = (2a_r+1)b_{n-1} - a_r^2 b_{n-2} = (2r^2+2r+1)b_{n-1} - r^2(r+1)^2 b_{n-2}. \tag{1}$$

Since $b_1 = r^2+r+1$ and $b_2 = r^4+2r^3+4r^2+3r+1$, we can define $b_0 = 1$ such that (1) is valid for $n \geq 2$. The characteristic equation $q^2 - (2r^2+2r+1)q + r^2(r+1)^2 = 0$ of the sequence $(b_n)_{n\geq 0}$ has roots r^2 and $(r+1)^2$. Hence,

$$b_n = Ar^{2n} + B(r+1)^{2n+1},$$

for some functions A and B of r that are independent of n. From the initial values $b_0 = 1$ and $b_1 = r^2+r+1$, we find $A = r/(2r+1)$ and $B = (r+1)/(2r+1)$. Therefore,

$$b_n = \frac{r^{2n+1} + (r+1)^{2n+1}}{2r+1}$$

for all integers $n \geq 1$ and $r \geq 0$.

Also solved Paul Bruckman and the proposer.

Another Fibonacci determinant

H-613 Proposed by Jayantibhai M. Patel, Ahmedabad, India (Vol. 42, no. 3, August 2004)

For any positive integer n, prove that

$$\begin{vmatrix} F_n^2 & -F_nF_{n+3} & F_{n+3}^2 & F_nF_{n+3} \\ -F_nF_{n+3} & F_{n+3}^2 & F_nF_{n+3} & F_n^2 \\ F_{n+3}^2 & F_nF_{n+3} & F_n^2 & -F_nF_{n+3} \\ F_nF_{n+3} & F_n^2 & -F_nF_{n+3} & F_{n+3}^2 \end{vmatrix} = -(2F_{2n+3})^4.$$

Solution by the editor, based on the solution by José Luis Díaz-Barrero, Barcelona, Spain

For convenience let $a = F_n$ and $b = F_{n+3}$. Then the given determinant becomes

$$\begin{vmatrix} a^2 & -ab & b^2 & ab \\ -ab & b^2 & ab & a^2 \\ b^2 & ab & a^2 & -ab \\ ab & a^2 & -ab & b^2 \end{vmatrix}. \tag{1}$$

It is wellknown that the value of this last symmetric determinant is $-(a^2+b^2)^4$ and this can be confirmed by an easy computation. Alternatively, by interchanging columns two and four, columns three and four, rows three and four, and then changing the sign of the third column and last row, we get that the value of the determinant equals

$$-\begin{vmatrix} a^2 & ab & ab & b^2 \\ -ab & a^2 & -b^2 & ab \\ ab & b^2 & -a^2 & -ab \\ -b^2 & ab & ab & -a^2 \end{vmatrix} = -|\boldsymbol{A} \otimes \boldsymbol{B}|,$$

where

$$\boldsymbol{A} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Since $\boldsymbol{A}$ and $\boldsymbol{B}$ are 2×2 matrices of determinants $\mp(a^2+b^2)$, and $|\boldsymbol{A} \otimes \boldsymbol{B}| = |\boldsymbol{A}|^2|\boldsymbol{B}|^2$, the claim about the value of the determinant (1) being $-(a^2+b^2)^4$ follows.

Finally, from [1] Exercise 30, p. 97, and Exercise 61, p. 97, we know that, for all positive integer n,

$$F_n^2 + F_{n+3}^2 = 2(F_{n+1}^2 + F_{n+2}^2) = 2F_{2n+3}.$$

[1] Thomas Koshy, Fibonacci and Lucas numbers with applications. Pure and Applied Mathematics. Wiley-Interscience, New York, 2001.

Also solved by Gurdial Arora, Paul Bruckman, Charles Cook, Kenneth Davenport, Harris Kwong and H.-J. Seiffert.

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