# ADVANCED PROBLEMS AND SOLUTIONS

*Edited by*
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*Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to FLORIAN LUCA, IMATE, UNAM, AP. POSTAL 61-3 (XANGARI), CP 58 089, MORELIA, MICHOACAN, MEXICO, or by e-mail at fluca@matmor.unam.mx as files of the type tex, dvi, ps, doc, html, pdf, etc. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions sent by regular mail should be submitted on separate signed sheets within two months after publication of the problems.*

## PROBLEMS PROPOSED IN THIS ISSUE

### H-658 Proposed by José Luis Díaz-Barrero, Barcelona, Spain

Let $n$ be a positive integer. Prove that

$$F_{2n} < \frac{1}{2}\Big(\frac{2^n F_n F_{n+1}}{F_{n+2}-1} + \binom{2n}{n}\frac{F_{n+2}-1}{2^n}\Big).$$

### H-659 Proposed by N. Gauthier, Kingston, Ontario

Let $m \geq 0$, $n \geq 0$, $a$, $b$, $q$ and $s$ be integers. Also let $\{c(m,k) = c(m,k;n,a,b) : 0 \leq k \leq m\}$ represent the set of coefficients that are given by the following recurrence, with the initial value and boundary conditions shown:

$$c(m+1,k) = (b+ka)c(m,k) + (n-k+1)ac(m,k-1);$$
$$c(0,0) = 1;\ c(m,-1) = c(m,m+1) = 0.$$

Also let $F_{n+2} = uF_{n+1} + F_n$ and $L_{n+2} = uL_{n+1} + L_n$ be the recurrences for the generalized Fibonacci and Lucas polynomials of order $n$, $F_n = F_n(u)$ and $L_n = L_n(u)$, respectively, where $F_0 = 0$, $F_1 = 1$, and $L_0 = 2$, $L_1 = u$. Prove that the following identities hold:

a) $\sum_{r=0}^{n} \binom{n}{r}(ar+b)^m = \sum_{k=0}^{m} 2^{n-k}c(m,k)$;

b) $\sum_{r=0}^{n}(-1)^{qa(n-r)}\binom{n}{r}(ar+b)^m F_{2qar+s} = \sum_{k=0}^{m} c(m,k)L_{qa}^{n-k}F_{qa(n+k)+s}$.

**H-660 Proposed by Emrah Kilic, Ankara, Turkey**

For positive integers $n$ and $k$ such that $n \geq 2k-1$, show that

$$\frac{1}{F_n} = \frac{1}{F_{n-k+1}F_{k+1}} + \frac{F_{n-k-1}F_{k-1}F_4}{F_{n-k+1}F_{n-k+2}F_{k+1}F_{k+2}} +$$

$$\frac{\left(\prod_{i=1}^{n-k} F_i^2\right)\left(\prod_{i=1}^{k} F_i^2\right)}{F_{n-k}F_k} \sum_{t=1}^{k-2} \left\{ \frac{F_{n-k-1-t}F_{k-1-t}F_{4+2t}}{\left(\prod_{i=1}^{n-k-1-t} F_i\right)\left(\prod_{i=1}^{n-k+2+t} F_i\right)\left(\prod_{i=1}^{k-1-t} F_i\right)\left(\prod_{i=1}^{k+2+t} F_i\right)} \right\}.$$

## SOLUTIONS

### A determinant with Fibonacci and Lucas numbers

**H-640 Proposed by Jayantibhai M. Patel, Ahmedabad, India (Vol. 44, No. 2, May 2006)**

For any positive integer $n \geq 2$, prove that the value of the following determinant

$$\begin{vmatrix} -L_{2n} & F_{2n} & L_n^2 & 2F_{2n} & L_n^2 \\ F_{2n} & -3(3F_n^2+2(-1)^n) & F_{2n} & 2F_n^2 & F_{2n} \\ L_n^2 & F_{2n} & -L_{2n} & 2F_{2n} & L_n^2 \\ 2F_{2n} & 2F_n^2 & 2F_{2n} & -6F_{n+1}F_{n-1} & 2F_{2n} \\ L_n^2 & F_{2n} & L_n^2 & 2F_{2n} & -L_{2n} \end{vmatrix}$$

is $(L_n^2 + L_{2n})^5$.

**Solution by Harris Kwong, SUNY Fredonia, Fredonia, NY**

Let $A$ denote the underlying matrix in the determinant, and define $\lambda = L_n^2 + L_{2n}$. Using the identities

$$5F_n^2 = L_{2n} - 2(-1)^n, \quad L_n^2 = L_{2n} + 2(-1)^n,$$

$$F_{n+1}F_{n-1} = F_n^2 + (-1)^n, \; 5F_{n+1}F_{n-1} = L_{2n} + 3(-1)^n,$$

we find

$$10F_n^2 + 6(-1)^n = 2L_{2n} + 2(-1)^n = 2L_{2n} + L_n^2 - L_{2n} = \lambda,$$

and

$$6F_{n+1}F_{n-1} = F_n^2 + L_{2n} + 4(-1)^n = F_n^2 + L_{2n} + L_n^2 - 5F_n^2 = \lambda - 4F_n^2.$$

Hence,

$$-3(3F_n^2 + 2(-1)^n) = -\lambda + F_n^2 \quad \text{and} \quad -6F_{n+1}F_{n-1} = -\lambda + 4F_n^2.$$

Consider

$$\boldsymbol{v}_1 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \; \boldsymbol{v}_2 = \begin{bmatrix} -F_n \\ L_n \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ L_n \\ -F_n \\ 0 \\ 0 \end{bmatrix}, \; \boldsymbol{v}_4 = \begin{bmatrix} 0 \\ L_n \\ 0 \\ 0 \\ -F_n \end{bmatrix}, \; \boldsymbol{v}_5 = \begin{bmatrix} L_n \\ F_n \\ L_n \\ 2F_n \\ L_n \end{bmatrix}.$$

Applying the identities listed above, we find $L_n^2 + 5F_n^2 = 2L_{2n}$ and $5F_n^2 + 3L_n^2 = 2\lambda$. Along with $F_nF_{2n} = F_n^2L_n$, $L_nF_{2n} = F_nL_n^2$, it is easy to verify that

$$A\boldsymbol{v}_i = -\lambda \boldsymbol{v}_i, \; i = 1,2,3,4, \text{ and } A\boldsymbol{v}_5 = \lambda \boldsymbol{v}_5.$$

Since $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4, \boldsymbol{v}_5$ are linearly independent, the determinant of $A$ is the product of its eigenvalues; hence, $\det(A) = \lambda^5$.

**Also solved by Paul S. Bruckman and E. Gokcen Kocer.**

## Fibonacci numbers count compositions

**H-641 Proposed by Emeric Deutsch, Polytechnic University, Brooklyn, NY (Vol. 44, No. 2, May 2006)**

A composition of $n$ is an ordered sequence of positive integers having sum equal to $n$. The terms of the sequence are called the parts of $n$. It is known that the number of compositions of $n$ with exactly $k$ parts is equal to $\binom{n-1}{k-1}$. Here, we consider a slightly modified concept assuming that there are two kinds of 1. Find:

**(i)** The number $a_n$ of compositions of $n$ (for example, $a_2 = 5$ because we have $(2)$, $(1,1)$, $(1,1')$, $(1',1)$, and $(1',1')$);

**(ii)** the number $c_{n,k}$ of compositions of $n$ with exactly $k$ parts (for example, $c_{4,2} = 5$ because we have $(1,3)$, $(1',3)$, $(3,1)$, $(3,1')$, and $(2,2)$).

**Solution by the proposer**

For part (i), note that we can generate all compositions of $n+1$ from the compositions of $n$ in the following manner:

(a) if the composition $\alpha$ of $n$ does not end in $1'$, then (i) we append 1 to $\alpha$, (ii) we append $1'$ to $\alpha$, and (iii) we increase the last part of $\alpha$ by unity.

(b) if the composition $\alpha$ of $n$ does end in $1'$, then (i) we append 1 to $\alpha$ and (ii) we append $1'$ to $\alpha$.

Clearly, the number of compositions that end with $1'$ is $a_{n-1}$. Consequently,

$$a_{n+1} = 3(a_n - a_{n-1}) + 2a_{n-1},$$

or $a_{n+2} = 3a_n - a_{n-1}$. Since $a_0 = 1$, $a_1 = 2$, we get easily that $a_n = F_{2n+1}$.

For part (ii), we first find the bivariate generating function $G(t,z)$ of the compositions with respect to the size of the composition (marked by $z$) and the number of parts (marked by $t$). In other words,

$$G(t,z) = \sum_{\substack{n\geq 0\\ k\geq 0}} c_{n,k} t^k z^n.$$

For this, let $S = \{1, 1', 2, 3, \dots\}$. Clearly, $S$ has generating function

$$2z + z^2 + z^3 + \cdots = z + \frac{z}{1-z}.$$

If we also want to mark the number of parts by $t$, then the generating function is

$$2tz + tz^2 + tz^3 + \cdots = t\Big(z + \frac{z}{1-z}\Big).$$

Thus, the generating function of the finite sequences of elements of $S$ is

$$\frac{1}{1-t(z+z/(1-z))} = \frac{1-z}{1-z-tz(2-z)}.$$

From here, by lengthy but elementary calculations one obtains that the coefficient of $t^k z^n$ in the above expansion is

$$c_{n,k} = \sum_{j=0}^{k} \binom{k}{j} \binom{n-k-1+j}{n-k}.$$

**Note.** Paul S. Bruckman and Ralph P. Grimaldi supplied equivalent expressions for $c_{n,k}$.

**Also solved by Paul S. Bruckman and Ralph P. Grimaldi.**

## An irrational limit

**H-642 Walther Janous, Innsbruck, Austria (Vol. 44, No. 2, May 2006)**

Determine the limit

$$\lim_{n\to\infty} \left( \frac{L_{n+2}^2}{F_{n+2}} - \sum_{k=1}^{n} \frac{L_k^2}{F_k} \right).$$

**Solution by H.-J. Seiffert, Berlin, Germany**

From equation ($I_{12}$) in [1], it follows that

$$\frac{L_k^2}{F_k} = 5F_k + 4\frac{(-1)^k}{F_k}$$

for all natural numbers $k$. Summing over $k = 1, 2, \ldots, n$ subtracting the resulting equation from the one obtained with $k = n+2$ and applying the known identity $\sum_{k=1}^{n} F_k = F_{n+2} - 1$, which is ($I_1$) in [1], one gets

$$\frac{L_{n+2}^2}{F_{n+2}} - \sum_{k=1}^{n} \frac{L_k^2}{F_k} = 5 + \frac{4(-1)^n}{F_{n+2}} - 4\sum_{k=1}^{n} \frac{(-1)^k}{F_k}.$$

It is seen that the limit $L$ in question equals

$$L = 5 - 4\sum_{k=1}^{\infty} \frac{(-1)^k}{F_k}.$$

According to [2], no simple closed form algebraic expression for the infinite sum occurring on the right hand side is known.

[1] V. E. Hoggatt, Jr. Fibonacci and Lucas numbers. Santa Clara, CA, The Fibonacci Association, 1979.
[2] S. Rabinowitz. "Algorithmic summation of reciprocals of products of Fibonacci numbers". The Fibonacci Quarterly **37.2** (1999):122–27.

**Also solved by Paul S. Bruckman.**

## Ranks of apparition of twin primes

**H-643 Proposed by John H. Jaroma, Loyola College in Maryland, Baltimore, MD (Vol. 44, No. 3, August 2006)**

The *rank of apparition* of a prime $p$ in $\{F_n\}$ is the index of the first term in the Fibonacci sequence that contains $p$ as a divisor. Furthermore, $p$ is said to have *maximal rank of apparition* provided that its rank of apparition in the underlying sequence is either $p-1$ or $p+1$. Recall that a pair of *twin primes* is a pair of consecutive odd integers $p$ and $p+2$ each of which is prime. Determine if both components of a pair of twin primes can simultaneously have maximal rank of apparition in $\{F_n\}$.

**Solution by the proposer**

Let $z(p)$ denote the rank of apparition of $p$ in $\{F_n\}_{n\geq 1}$. Since $z(5)=5\neq 5\pm 1$, it is without loss of generality to assume that the twin pair in question is $p,\ p+2$ for some prime $p>5$. Let $\left(\frac{a}{p}\right)$ denote the Legendre symbol of $a$ with respect to $p$. It is known that $z(p)\mid (p-(5/p))/2$ if and only if $\left(\frac{-1}{p}\right)=1$. Clearly, one of $p$ and $p+2$ is congruent to 1 (mod 4), therefore either $\left(\frac{-1}{p}\right)=1$ or $\left(\frac{-1}{p+2}\right)=1$, which shows that not both $p$ and $p+2$ can have maximal rank of apparition.

**Also solved by Paul S. Bruckman and H.-J. Seiffert.**

**Errata.** In H-649, the right hand side of the equation "$=\sec F_c$" should be "$=F_c$". The Editor apologizes for the oversight.

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