ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by September 15, 2008. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \ F_0 = 0, \ F_1 = 1;$$

$$L_{n+2} = L_{n+1} + L_n, \ L_0 = 2, \ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-1036 Proposed by José Luis Díaz-Barrero and Miquel Grau-Sánchez, Universitat Politécnica de Catalunya, Barcelona, Spain

Let n be a positive integer. Prove that

$$\frac{1}{F_{n+2}} + \sum_{k=1}^{n} \left(\frac{1}{F_k^2} + \frac{1}{F_{k+1}^2} \right)^{1/2} < 1 + \sum_{k=1}^{n} \frac{1}{F_k}.$$

B-1037 Proposed by Vladimir Pletser, The Netherlands

Let p be a prime. Prove or disprove each of the following:

(a) Except for $p = 5$, $p = 4r + 1$ divides $(F_{2r})(F_{2r+1})$.

(b) $p = 4r + 3$ divides $(L_{2r+1})(L_{2r+2})$.

B-1038 Proposed by the Problem Editor

Prove or disprove:

$$F_{n+1}^4 + F_n^4 + 2F_n^3 + F_n^2 - F_{n+1}^2 - 2F_{n+1}^2 F_n - 2F_{n+1}^2 F_n^2$$

is divisible by F_{n+2} and F_{n-1} for all integers $n \geq 1$.

B-1039 Proposed by Pantelimon George Popescu, Bucuresti, Romania and José Luis Díaz-Barrero, Barcelona, Spain

Let n be a positive integer. Prove that

$$\frac{1}{F_1F_3} + \frac{1}{F_2F_4} + \cdots + \frac{1}{F_nF_{n+2}} \geq \frac{2n^2}{F_nF_{n+1} + F_{n+2}^2 - 1}.$$

SOLUTIONS

A Difference of Two Geometric Means

B-1024 Proposed by José Luis Díaz-Barrero and Miquel Grau-Sánchez, Universidad Politécnica de Cataluña, Barcelona, Spain (Vol. 44, no. 4, November 2006)

Let n be a positive integer. Prove that

$$F_{n-1} \le \frac{1}{2}\left(\sqrt[n]{\prod_{k=1}^{n}(k+L_n)} - \sqrt[n]{\prod_{k=1}^{n}(k+F_n)}\right).$$

Solution by Paul S. Bruckman, Sointula, BC V0N 3E0 (Canada)

Let

$$G_n = \sqrt[n]{\prod_{k=1}^{n}(k+L_n)}, H_n = \sqrt[n]{\prod_{k=1}^{n}(k+F_n)},\ n = 1, 2, \cdots.$$

Note that G_n is the geometric mean of the quantities $(1+L_n), (2+L_n), \cdots, (n+L_n)$, while H_n is the geometric mean of the quantities $(1+F_n), (2+F_n), \cdots, (n+F_n)$; since $L_n \ge F_n$ for all $n \ge 1$, it folllows that $G_n \ge H_n$. Also, it is clear that $G_n \ge L_n, H_n \ge F_n$.

We may state the following: $G_n = L_n + A_n$, say, where $1 \le A_n \le n$. Similarly, $H_n = F_n + B_n$, say, where $1 \le B_n \le n$. Moreover, since $L_n \ge F_n$, it follows that $A_n \ge B_n$.

Therefore, $G_n - H_n = L_n - F_n + A_n - B_n \ge L_n - F_n = 2F_{n-1}$.

Equivalently, $F_{n-1} \le \frac{1}{2}(G_n - H_n)$. Q.E.D.

Also solved by Russell J. Hendel, H.-J. Seiffert, and the proposer.

An Exponential Inequality

B-1025 Proposed by José Luis Díaz-Barrero, Barcelona Spain and Pantelimon George Popescu, Bucharest, Romania (Vol. 44, no. 4, November 2006)

Let n be a positive integer. Prove that

$$\prod_{k=1}^{n} F_k^{(F_k+F_{2n})C(n,k)} \le \left(\prod_{k=1}^{n} F_k^{F_k C(n,k)}\right)^{2^n}$$

where $C(n,k)$, for all $k \in \{1, 2, \ldots, n\}$, is the binomial coefficient $\binom{n}{k}$.

Solution by H.-J. Seiffert, Thorwaldsenstr. 13, D-12157 Berlin

The sequences $(F_k)_{k=1}^n$ and $(\ln F_k)_{k=1}^n$, where ln denotes the natural logarithm, are both nondecreasing. Hence,

$$0 \le \sum_{j=1}^{n}\sum_{k=1}^{n}(F_j - F_k)(\ln F_j - \ln F_k)c(n,j)c(n,k),$$

or, equivalently,

$$\left(\sum_{k=1}^{n} F_k c(n,k)\right)\left(\sum_{k=1}^{n}(\ln F_k)c(n,k)\right) \le \left(\sum_{k=1}^{n} c(n,k)\right)\left(\sum_{k=1}^{n} F_k(\ln f_k)c(n,k)\right).$$

Since (see, for example, P. Haukkanen. "On a Binomial Sum for the Fibonacci and Related Numbers." *The Fibonacci Quarterly* **34.4** (1996): 326-31, (1) and (2)) $\sum_{k=1}^{n} F_k c(n,k) = F_{2n}$ and, by the Binomial Theorem, $\sum_{k=1}^{n} c(n,k) = 2^n - 1$, from the above inequality, one easily obtains the logarithmic form of the requested inequality.

Also solved by Paul S. Bruckman, Russell J. Hendel, and the proposer.

Area of Regular Pentagon

B-1026 Proposed by Steve Edwards, Southern Polytechnic State University, Marietta, GA
(Vol. 45, no. 1, February 2007)

Show that the area of a regular pentagon with side s is

$$\frac{s^2(\alpha+2)^{3/2}}{4}. \tag{1}$$

Solution by Scott H. Brown, Auburn University, Montgomery, AL

We found the following formula for the area of a regular pentagon from [1]

$$A = \frac{s^2\left(\sqrt{25+10\sqrt{5}}\right)}{4}. \tag{2}$$

We need to show

$$(\alpha+2)^{\frac{3}{2}} = (25+10\sqrt{5})^{\frac{1}{2}} \tag{3}$$

or more specific

$$(\alpha+2)^3 = (25+10\sqrt{5}). \tag{4}$$

Substituting $\alpha = \frac{1+\sqrt{5}}{2}$ into the left-hand side of (4) and expanding we have

$$\frac{1}{8}(5+\sqrt{5})^3 = \frac{1}{8}(200+80\sqrt{5}) = 25+10\sqrt{5} \tag{5}$$

Therefore (1) holds true.

[1] http://mathworld.wolfram.com/pentagon.html

Also solved by Michael R. Bacon and Charles K. Cook (jointly), Brian D. Beasley, Paul S. Bruckman, Kenneth B. Davenport, José Luis Diaz-Barrero and José Gibergans-Báguena (jointly), Russell J. Hendel, Harris Kwong, H.-J. Seiffert, David Stone and John Hawkins (jointly), and the proposer.

A Constant Ratio

B-1027 Proposed by M. N. Deshpande, Nagpur, India
(Vol. 45, no. 1, February 2007)

Define $\{x_n\}$ by $x_1 = 1, x_2 = 10$ and $x_{n+2} = 6x_{n+1} - x_n + 2$ for $n \geq 1$. Let P_n be the n^{th} Pell number for $n \geq 0$.

Prove or disprove:

$$\frac{8x_n(x_n+1)+20}{(P_{2n}-P_{2n-2})^2}$$

is a constant for all integers $n \geq 1$.

Solution by H.- J. Seiffert, Berlin, Germany

The sequence $(Q_n)_{n\geq 0}$ of the Pell-Lucas numbers satisfies the same recurrence as the sequence of the Pell numbers, but with initial conditions $Q_0 = Q_1 = 2$. Based on the recurrences (see [1], eqns. (3.22) and (3.23)) $P_{2n+4} = 6P_{2n+2} - P_{2n}$ and $Q_{2n+4} = 6Q_{2n+2} - Q_{2n}$, a simple induction argument shows that

$$x_n = 3P_{2n} - \frac{3}{4}Q_{2n} - \frac{1}{2}, \quad n \geq 1,$$

so that, by $P_{2n} - P_{2n-2} = 2P_{2n-1}$,

$$q_n = \frac{9((4P_{2n}) - Q_{2n})^2 + 4)}{8P_{2n-1}^2}, \quad n \geq 1,$$

is the quotient under consideration. Eqn. (3.25) of [1] implies that $4P_{2n} - Q_{2n} = (Q_{2n+1} + Q_{2n-1})/2 - Q_{2n} = Q_{2n-1}$. From eqn. (3.19) of [1], one finds that $Q_{2n-1}^2 + 4 = 8P_{2n-1}^2$. It follows that $q_n = 9$ for all $n \geq 1$.

Reference

1. A.F. Horadam & Bro. J.M. Mahon. "Pell and Pell-Lucas Polynomials." *The Fibonacci Quarterly* **23.1** (1985): 7-20.

Also solved by Paul S. Bruckman, Charles K. Cook, Kenneth B. Davenport, G. C. Greubel, Harris Kwong, David Stone and John Hawkins (jointly), and the proposer.

No Solution for this Equation!

B-1028 Proposed by Paul S. Bruckman, Sointula, Canada (Vol. 45, no. 1, February 2007)

Prove that the Diophantine equation

$$(a + b\alpha)^2 + (a + b\beta)^2 = c^2$$

has no solution in the positive integers a, b and c.

Solution by Brian D. Beasley, Department of Mathematics, Presbyterian College, Clinton, SC 29325

Using $\alpha + \beta = 1$ and $\alpha^2 + \beta^2 = 3$, we note that the given equation is equivalent to

$$2a^2 + 2ab + 3b^2 = c^2.$$

For contradiction, let k be the largest non-negative integer such that 2^k divides each of the positive integers a, b, and c. Then $a = 2^k a_1, b = 2^k b_1$, and $c = 2^k c_1$, where at least one of the positive integers a_1, b_1, or c_1 is odd. This yields

$$2a_1^2 + 2a_1b_1 + 3b_1^2 = c_1^2.$$

But if both a_1 and b_1 are odd, then the left-hand side is equivalent to 3 modulo 4, which is impossible for a square. Similarly, if a_1 is even and b_1 is odd, then we again obtain 3 modulo 4, while if a_1 is odd and b_1 is even, then we obtain 2 modulo 4, also impossible. Finally, if both a_1 and b_1 are even, then the left-hand side is even, which contradicts the fact that c_1 must be odd.

Also solved by Russell J. Hendel, Harris Kwong, H.-J. Seiffert, David Stone and John Hawkins (jointly), Konstantine Zelator, and the proposer.

We would like to belatedly acknowledge Russell J. Hendel and Kenneth Davenport as solvers for Problems B-1021, B-1022, and B-1023.