16 EXPANSION OF ANALYTIC FUNCTIONS IN POLYNOMIALS ASSOCIATED WITH FIBONACCI NUMBERS

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1. Introduction. A problem which has long been of fundamental interest in classical analysis is the expansion of a given function f(x) in a series of the form

$$(1.1) \qquad f(x) = \sum_{n=0}^{\infty} b_n P_n(x)$$

where $\{ P_n(x) \}$ is a prescribed sequence of polynomials, and where the coefficients b_n are numbers related to f. In particular, the innumerable investigations on expansions of "arbitrary" functions in orthogonal polynomials have led to many important convergence and summability theorems, and to various interesting results in the theory of approximation. (See, for example, Alexits [1], Szegö [2], Rainville [3], and Jackson [4].) Numerous recent studies have also been made on the expansion of analytic functions employing more general sets of polynomials (e.g., see Whittaker [5], or Boas and Buck [6]). There is thus already in existence a great wealth of theory which may be applied when a particular set of polynomials is introduced to accomplish a certain purpose.

In the present article, we shall apply some available results in order to consider the expansion of analytic functions in a series of a certain set of polynomials which can be associated with the famous numbers of Fibonacci. Our primary objective is to illustrate a simple, general technique that may be used to obtain expansions of a given class of functions in terms involving Fibonacci numbers. Some important broad questions and problems concerning convergence and the representability of our polynomial expansions in general will not be discussed, however.

2. Fibonacci Polynomials. By 'Fibonacci polynomials' we shall mean the sequence of polynomials $\{ \varphi_k(x) \}$,

($k = 0, 1, \ldots\ldots$) satisfying the recurrence relation[1]

$$\varphi_{k+2}(x) - 2x\,\varphi_{k+1}(x) - \varphi_k(x) = 0, \quad -\infty < x < \infty \tag{2.1}$$

with initial conditions

$$\varphi_0(x) = 0, \quad \varphi_1(x) = 1. \tag{2.2}$$

In the special case when $x = 1/2$, equations (2.1) and (2.2) clearly reduce to the well-known relations [7] that furnish the Fibonacci numbers $0, 1, 1, 2, 3, \ldots\ldots$, which we shall denote by $\varphi_k(1/2)$ or F_k.

A generating function defining the polynomials $\varphi_k(x)$ is

$$\frac{s}{1 - 2xs - s^2} = \sum_{k=0}^{\infty} \varphi_k(x)\, s^k . \tag{2.3}$$

Now since the left member of (2.3) changes sign if x is replaced by $(-x)$ and s by $(-s)$, we have

$$\varphi_k(-x) = (-1)^{k+1}\,\varphi_k(x) \tag{2.4}$$

thereby showing that $\varphi_{k+1}(x)$ is an odd function of x for k odd and an even function of x for k even. Upon expanding the left side of (2.3) and equating coefficients in s, we obtain the explicit formula

$$\varphi_{k+1}(x) = \sum_{m=0}^{[k/2]} \binom{k-m}{m} (2x)^{k-2m}, \quad (k \geq 0), \tag{2.5}$$

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[1] A related set of polynomials, which satisfies the recurrence relation $y_{k+2}(x) - x\,y_{k+1}(x) - y_k(x) = 0$, was considered in 1883 by Catalan [8]. The name 'Fibonacci polynomials' is also given to solutions of the relation $z_{k+2}(x) = z_{k+1}(x) + x\,z_k(x)$, $z_0(x) = 0$, $z_1(x) = 1$, investigated by Jacobsthal [9].

where $[k/2]$ is the greatest integer $\leq k/2$.

An alternative form for expressing the polynomials $\varphi_k(x)$ may be found by introducing the exponential generating function defined by

$$Y(s,x) = \sum_{k=0}^{\infty} \varphi_k(x) \frac{s^k}{k!} \tag{2.6}$$

This transforms the recurrence relation (2.1), and the initial conditions (2.2), into the differential equation

$$\frac{d^2Y}{ds^2} - 2x\frac{dY}{ds} - Y = 0 \tag{2.7}$$

with conditions

$$Y(0,x) = 0, \quad \left.\frac{dY}{ds}\right|_{s=0} = 1. \tag{2.8}$$

The solution of (2.7) thus yields the generating function

$$Y(s,x) = \frac{1}{2\sqrt{(1+x^2)}} [e^{sa_1} - e^{sa_2}], \tag{2.9}$$

where

$$a_1 = x + \sqrt{(1+x^2)}, \quad a_2 = x - \sqrt{(1+x^2)}. \tag{2.10}$$

If we now apply the inverse transform

$$\varphi_k(x) = \left.\frac{d^kY}{ds^k}\right|_{s=0}, \quad k = 0,1,2,\ldots. \tag{2.11}$$

$$x = \sinh\omega, \quad \sqrt{(1+x^2)} = \cosh\omega \tag{2.12}$$

we obtain

$$\varphi_{2k}(x) = \frac{\sinh 2k\omega}{\cosh \omega}$$

(2.13) $(k = 0, 1, 2, \ldots.)$

$$\varphi_{2k+1}(x) = \frac{\cosh (2k+1)\omega}{\cosh \omega}$$

3. Some Other Relations. We note, as can easily be shown that the polynomials $\varphi_m(x)$ are related to Chebyshev's polynomials $U_m(x)$ of the second kind[2] [3] by

(3.1) $\varphi_0(x) = U_0(ix) = 0, \; \varphi_{m+1}(x) = (-i)^m \, U_{m+1}(ix),$

$(i = \sqrt{-1}, \; m \geq 0).$

The Chebyshev polynomials themselves of course belong to a larger family designated as 'ultraspherical polynomials' or sometimes 'Gegenbauer polynomials' 2]. Unlike those of Chebyshev or of Gegenbauer, however, our Fibonacci polynomials $\varphi_m(x)$ are not orthogonal on any interval of the real axis.

The sequence $\varphi_{k+1}(x)$, $(k = 0, 1, 2, \ldots)$ is a so-called simple set, since the polynomials are of degree precisely k in x, as is seen from (2.5). Thus the linearly independent set contains one polynomial of each degree, and any polynomial $P_n(x)$ of degree n can clearly be expressed linearly in terms of the elements of the basic set; that is, there always exist constants c_k such that the finite sum

(3.2) $$P_n(x) = \sum_{k=0}^{n} c_k \, \varphi_{k+1}(x)$$

is a unique representation of $P_n(x)$.

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[2]These polynomials of Chebyshev are not to be confused with the Chebyshev polynomials $T_m(x)$ of the first kind, which are useful in optimal-interval interpolation[10].

Before we seek the explicit expression for the coefficients in the expansion of a given analytic function f(x) in series of our basic set $\{\varphi_{k+1}(x)\}$, it is useful to have x^n in a series of this set. Taking Fibonacci polynomials as defined by formula (2.5), we thus need the easily established reciprocal relation(3),

$$(3.3)\qquad x^n = (1/2^n) \sum_{r=0}^{[n/2]} (-1)^r \binom{n}{r} \frac{n-2r+1}{n-r+1} \varphi_{n+1-2r}(x), \quad n \geq 0,$$

which could also be re-arranged in the form

$$(3.4)\qquad x^n = \sum_{j=0}^{n} \gamma_{nj} \varphi_{j+1}(x)$$

that will then contain only even φ's when n is odd, and odd φ's when n is even.

4. Expansion of Analytic Functions. We assume that our arbitrarily given function f(x) can be represented by a power series

$$(4.1)\qquad f(x) = \sum_{n=0}^{\infty} a_n x^n$$

having a radius of convergence of $\rho \geq 1/2$, with the coefficients a_n expressed by

$$(4.2)\qquad a_n = \frac{f^{(n)}(0)}{n!} \qquad (n = 0, 1, \ldots)$$

Formal substitution of relation (3.3) into (4.1) yields the desired polynomial expansion

$$(4.3)\qquad f(x) \asymp \sum_{k=0}^{\infty} c_k \varphi_{k+1}(x),$$

(3) In view of (2.12) and (2.13), this relation is an equivalent form for known expressions for powers of the hyperbolic function sinh ω.

where the coefficients are finally determined from the formula

$$c_k = (k+1)\sum_{j=0}^{\infty} \frac{(-1)^j a_{2j+k}}{2^{2j+k}(j+k+1)} \binom{2j+k}{j} \tag{4.4}$$

Convergence properties of the general basic series (1.1) have been investigated by Whittaker [5], by Boas and Buck [6], and by others. If Whittaker's results are applied to our case, it can be shown that the expansion (4.3) will converge absolutely and uniformly to the function f(x) in $|x| \leq \zeta$ if the series

$$\sum_{n=0}^{\infty} |a_n| V_n(\zeta) \tag{4.5}$$

converges, where $V_n(\zeta)$ is given by

$$V_n(\zeta) = \sum_{j=0}^{\infty} |\gamma_{nj}| M_j(\zeta), \tag{4.6}$$

with

$$M_j(\zeta) = \underset{|x|=\zeta}{\operatorname{Max}} |\varphi_{j+1}(x)|, \tag{4.7}$$

and with γ_{nj} being the coefficients in (3.3) after they have been re-arranged in the form (3.4).

Now, we may also introduce a parameter 2α such that $|2\alpha x| \leq \zeta$, and may thus start with the form

$$f(2\alpha x) = \sum_{n=0}^{\infty} (2^n \alpha^n a_n) x^n = \sum_{n=0}^{\infty} A_n x^n, \tag{4.8}$$

where

(4.9) $$A_n = \frac{1}{n!} \frac{d^n}{dx^n} f(2\alpha x) \Big|_{x=0}$$

The expansion (4.3) in terms of Fibonacci polynomials then becomes

(4.10) $$f(2\alpha x) = \sum_{k=0}^{\infty} \beta_k \varphi_{k+1}(x) ,$$

with the coefficients β_k now being determined by the equation

(4.11) $$\beta_k(\alpha) = (k+1) \sum_{j=0}^{\infty} \frac{(-1)^j \alpha^{2j+k}}{j+k+1} a_{2j+k} \binom{2j+k}{j}$$

For our purposes, the form (4.10) is often more convenient than that of (4.3).

If we take $x = 1/2$, the polynomials $\varphi_k(x)$ become the numbers of Fibonacci, $\varphi_k(1/2) = F_k$, so that the series

(4.12) $$f(\alpha) = \sum_{k=0}^{\infty} \beta_k(\alpha) \varphi_{k+1}(1/2) = \sum_{k=0}^{\infty} \beta_k(\alpha) F_{k+1}$$

furnishes a formal expansion of the function $f(\alpha)$ in terms involving Fibonacci numbers. One apparent use of the series expansion (4.12) is for the case in which it is desired to make a given analytic function f serve as a generating function of the Fibonacci-number sequence.

5. Examples. We first consider the function

(5.1) $$f(x) = e^{2\alpha x} , \qquad (0 < |\alpha| < \infty) ,$$

where

(5.2) $$a_n = 2^n \alpha^n / n! .$$

The coefficients c_k in (4.4) are then given by the formula

$$(5.3) \qquad c_k = (k+1) \sum_{j=0}^{\infty} \frac{(-1)^j \alpha^{2j+k}}{(2j+k)!\,(j+k+1)} \binom{2j+k}{j}$$

or finally by

$$(5.4) \qquad c_k = \frac{k+1}{\alpha} J_{k+1}(2\alpha), \qquad (k = 0, 1, 2, \ldots\ldots)$$

where J_{k+1} is Bessel's function [11] of order $k+1$. The polynomial expansion (4.3) therefore yields formally

$$(5.5) \quad e^{2\alpha x} = (1/\alpha) \sum_{k=0}^{\infty} (k+1)\, J_{k+1}(2\alpha)\, \varphi_{k+1}(x)$$

$$= (1/\alpha) \sum_{m=1}^{\infty} m\, J_m(2\alpha)\, \varphi_m(x)$$

We note that

$$(5.6) \quad \lim_{m \to \infty} \frac{(m+1)\, J_{m+1}(2\alpha)\, \varphi_{m+1}(x)}{m\, J_m(2\alpha)\, \varphi_m(x)}$$

$$= \lim_{m \to \infty} \frac{(x + \sqrt{1 + x^2})}{m}\, \alpha = 0,$$

so that the series (5.5) is convergent for all finite values of x if the parameter α remains also finite.

From (5.5), with the relations

$$(5.7) \qquad \begin{aligned} \cosh 2\alpha x &= (e^{2\alpha x} + e^{-2\alpha x})/2, \\ \sinh 2\alpha x &= (e^{2\alpha x} - e^{-2\alpha x})/2, \end{aligned}$$

we immediately obtain the two expansions

$$\cosh 2\alpha x = (1/\alpha) \sum_{m=1}^{\infty} (2m-1)\, J_{2m-1}(2\alpha)\, \varphi_{2m-1}(x) \tag{5.8}$$

and

$$\sinh 2\alpha x = (1/\alpha) \sum_{m=1}^{\infty} 2m\, J_{2m}(2\alpha)\, \varphi_{2m}(x) \tag{5.9}$$

Similarly, we have

$$\cos 2\alpha x = (1/\alpha) \sum_{m=1}^{\infty} (-1)^{m+1} (2m-1)\, I_{2m-1}(2\alpha)\, \varphi_{2m-1}(x) \tag{5.10}$$

and

$$\sin 2\alpha x = (1/\alpha) \sum_{m=1}^{\infty} (-1)^{m+1} (2m)\, I_{2m}(2\alpha)\, \varphi_{2m}(x), \tag{5.11}$$

where I_k is the modified Bessel function [11] of the first kind of order k.

To produce expansions involving the Fibonacci numbers F_k. we simply set $x = 1/2$. Hence from (5.5), (5.8), (5.9), (5.10) and (5.11), it is seen for $0 < |\alpha| < \infty$ that

(5.12)

$$e^{\alpha} = (1/\alpha) \sum_{m=1}^{\infty} m\, J_m(2\alpha)\, F_m$$

$$\cosh \alpha = (1/\alpha) \sum_{m=1}^{\infty} (2m-1)\, J_{2m-1}(2\alpha)\, F_{2m-1}$$

$$\sinh \alpha = (1/\alpha) \sum_{m=1}^{\infty} (2m)\, J_{2m}(2\alpha)\, F_{2m}$$

(5.12)

$$\cos \alpha = (1/\alpha) \sum_{m=1}^{\infty} (-1)^{m+1} (2m-1) I_{2m-1}(2\alpha) F_{2m-1}$$

$$\sin \alpha = (1/\alpha) \sum_{m=1}^{\infty} (-1)^{m+1} (2m) I_{2m}(2\alpha) F_{2m}.$$

As $\alpha \longrightarrow 0$, the right-hand sides of (5.12) all become indeterminate forms, but the correct result is obtained in the limit. The particular series expansions (5. 5), (5. 8), (5. 9), (5.10), (5.11) and (5.12) are apparently not found in the literature in the specific form we have presented for our purposes; they could be related, however, to some expansions due to Gegenbauer [11, page 369].

Many higher transcendental functions can also be explicitly developed along similar lines. For instance, without difficulty we may derive the series expansions

$$I_1(\alpha) = (2/\alpha) \sum_{m=1}^{\infty} m\, J_m^2(\alpha) F_{2m}$$

(5.13)
$$J_1(\alpha) = (2/\alpha) \sum_{m=1}^{\infty} (-1)^{m+1} m\, I_m^2(\alpha) F_{2m}$$

for the Bessel functions I_1 and J_1.

The coefficients in the above examples all involve Bessel's functions, but this indeed would not be the case in general. For instance, for

$$|2\alpha x| < 1$$

we can show from (4.10) and (4.11) that

(5.14)
$$\ln (1 + 2\alpha x) = -[r^2/2 + \ln r/\alpha]\, \varphi_1(x) + \sum_{k=1}^{\infty} (-1)^{k+1} r^k [1/k + r^2/(k+2)]\, \varphi_{k+1}(x),$$

where

$$r = \frac{\sqrt{(1 + 4\alpha^2)} - 1}{2\alpha} . \tag{5.15}$$

With x = 1/2, we then have, for $|\alpha| < 1$,

$$\ln(1+\alpha)\ (r/\alpha) = -(r^2/2)\, F_1 + \sum_{k=1}^{\infty} (-1)^{k+1} r^k [1/k + r^2/(k+2)]\, F_{k+1}. \tag{5.16}$$

6. Another Approach.[4] The coefficients β_k in our basic series expansion (4.10) or (4.12) may be obtained by an alternative procedure which is based on relations (3.1) and certain known properties of the orthogonal polynomials $U_k(x)$. (A good reference giving many properties of U_k is [12]).

If our prescribed function $f(2\alpha x)$ can be expanded in the formal series

$$f(2\alpha x) = \sum_{k=0}^{\infty} b_k U_{k+1}(x), \quad |x| < 1, \quad |2\alpha x| \leq \zeta \tag{6.1}$$

the coefficients b_k are given by

$$b_k = (2/\pi)\int_{-1}^{1} f(2\alpha x)\sqrt{(1-x^2)}\,U_{k+1}(x)\,dx, \quad (k = 0, 1 \ldots). \tag{6.2}$$

With the relations

$$x = \cos v, \quad U_k(x) = (\sin kv)/\sqrt{(1-x^2)} \tag{6.3}$$

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[4] We could also apply the tools employed in [6] but have written this paper without assuming knowledge of complex-variable methods.

the expression (6. 2) becomes

(6. 4)
$$b_k(\alpha) = \frac{1}{\pi}\int_0^{\pi} f(2\alpha \cos v) \; [\cos kv - \cos(k+2)v] \, dv.$$

In view of relations (3.1, (4.10) and (6.1), we then have formally,

$$(6.5) \quad \beta_k(\alpha) = \frac{i^k}{\pi}\int_0^{\pi} f(-2\alpha\, i\cos v) \; [\cos v - \cos(k+2)v] \, dv$$

as an equation for β_k in integral form.

In the special case when

$$(6.6) \qquad f(2\alpha x) = e^{2\alpha x}$$

we find

$$(6.7) \quad \beta_k(\alpha) = \frac{i^k}{\pi}\int_0^{\pi} e^{-2\alpha i \cos v}[\cos kv - \cos(k+2)v] \, dv$$

$$= J_k(2\alpha) + J_{k+2}(2\alpha) = \frac{k+1}{\alpha} J_{k+1}(2\alpha),$$

which is the same result obtained in example (5. 5). Usually, however, the integrals (6. 5) involving a given function f are not available, so that the expression (4.11) is more often the better procedure for determining the coefficients β_k.

The particular expansions (5.12) and (5.13), or (4.12) in general, turn out to have little use as a means of obtaining efficient approximations for computational purposes. Independent of numerical or physical applications, however, the introduction of Fibonacci numbers into various expressions involving classical functions has a certain interest and fascination in itself.

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PROBLEM DEPARTMENT

P-1. Verify that the polynomials $\varphi_{k+1}(x)$ satisfy the differential equation

$$(1+x^3)\,y'' + 3xy' - k(k+2)y = 0 \qquad (k = 0, 1, 2, \ldots)$$

P-2. Derive the series expansion