A PRIMER ON THE FIBONACCI SEQUENCE
PART I

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1. INTRODUCTION.

The proofs of Fibonacci identities serve as very suitable examples of certain techniques encountered in a first course in algebra. With this in mind, it is the intention of this series of articles to introduce the beginner to a few techniques in proving some number theoretic identities as well as furnishing examples of well-known methods of proof such as mathematical induction. The collection of proofs that will be given in this series may serve as a source of elementary examples for classroom use.

The use of matrix algebra in proving many theorems will be developed from basic principles in the next issue.

2. SOME SIMPLE PROPERTIES OF THE FIBONACCI SEQUENCE.

By observation of the sequence $\{1, 1, 2, 3, 5, 8, \ldots\}$, it is easily seen that each term is the sum of the two preceding terms. In mathematical language, we define this sequence by letting

$F_1 = 1$, $F_2 = 1$, and, for all integral n,

Definition (A) $\qquad F_{n+2} = F_{n+1} + F_n$ holds.

The first few Fibonacci numbers are:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, ...

The Lucas Numbers, L_n, satisfy the same recurrence relation but have different starting values, namely,

$L_1 = 1$, $L_2 = 3$, and

Definition (B) $\qquad L_{n+2} = L_{n+1} + L_n$ holds.

The first few Lucas Numbers are:

1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, ...

The following are some simple formulas which are called Fibonacci Number Identities or Lucas Number Identities for $n \geq 1$.

I. $F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$

II. $L_1 + L_2 + L_3 + \cdots + L_n = L_{n+2} - 3$

III. $F_{n+1} F_{n-1} - F_n^2 = (-1)^n$

IV. $L_{n+1} L_{n-1} - L_n^2 = 5(-1)^{n+1}$

V. $L_n = F_{n+1} + F_{n-1}$

VI. $F_{2n+1} = F_{n+1}^2 + F_n^2$

VII. $F_{2n} = F_{n+1}^2 - F_{n-1}^2$

VIII. $F_{2n} = F_n L_n$

IX. $F_{n+p+1} = F_{n+1} F_{p+1} + F_n F_p$

X. $F_1^2 + F_2^2 + \cdots F_n^2 = F_n F_{n+1}$

XI. $L_n^2 - 5 F_n^2 = 4(-1)^n$

XII. $F_{-n} = (-1)^{n+1} F_n$

XIII. $L_{-n} = (-1)^n L_n$

3. MATHEMATICAL INDUCTION

Any proofs of the foregoing identities ultimately depend upon the postulate of complete mathematical induction.

First one has a formula involving an integer n. For some values of n the formula has been seen to be true. This may be one, two, or say, twenty times. Now the excitement sets in....Is it true for all positive n? One may prove this by appealing to mathematical induction, whose three phases are:

A. Statement P(1) is true by trial. (If you can't find a first true case...why do you think it's true for any n let alone all n? Here you need some true cases to start with.)

An example of statement P(n) is

$$1 + 2 + 3 + \ldots\ldots + n = n(n+1)/2.$$

It is simple to see P(1) is true, that is

$$1 = 1(1+1)/2.$$

B. The truth of statement P(k) logically implies the truth of statement P(k + 1). In other words: If P(k) is true, then P(k + 1) is true. This step is commonly referred to as the inductive transition.

The actual method used to prove this implication may vary from simple algebra to very profound theory.

C. The statement that 'The proof is complete by mathematical induction.'

4. SOME ELEMENTARY FIBONACCI PROOFS.

Let us prove identity I.
Recall from (A) that $F_1 = 1$, $F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$

Statement P(n) is

$$F_1 + F_2 + F_3 + \dots + F_n = F_{n+2} - 1$$

A. P(1) is true, since $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, so that

$$F_1 = 1 = 2 - 1 = F_3 - 1.$$

B. Assume P(k) is true, that is

$$F_1 + F_2 + \dots + F_k = F_{k+2} - 1$$

From this we will show that the truth of P(k) demands the truth of P(k + 1), which is

$$(F_1 + F_2 + \dots + F_k) + F_{k+1} = F_{k+3} - 1.$$

Since we assume P(k) is true, we may therefore assume that, in P(k + 1), we may replace $(F_1 + F_2 + \dots + F_k)$ by $(F_{k+2} - 1)$. That is, P(k + 1) may be rewritten equivalently as $(F_{k+2} - 1) + F_{k+1} = (F_{k+2} + F_{k+1}) - 1 = F_{k+3} - 1$. This is now clearly true from (A), which for n = k + 1 becomes $F_{k+3} = F_{k+2} + F_{k+1}$.

C. The proof is complete by mathematical induction.

5. A BIT OF THEORY (Cramer's Rule)

Given a second order determinant,

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

we have,

THEOREM I. $$\begin{vmatrix} ax & b \\ cx & d \end{vmatrix} = x\,D, \quad -\infty < x < \infty .$$

Proof: By definition

$$\begin{vmatrix} ax & b \\ cx & d \end{vmatrix} = (ax)\,d - b(cx) = x(ad-bc) = xD .$$

THEOREM II.

$$\begin{vmatrix} ax + by & b \\ cx + dy & d \end{vmatrix} = \begin{vmatrix} ax & b \\ cx & d \end{vmatrix} = xD, \quad \begin{matrix} -\infty < y < \infty, \\ -\infty < x < \infty . \end{matrix}$$

Proof: This is a simple exercise in algebra.

Suppose a system of two simultaneous equations possesses a unique solution (x_0, y_0), that is

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

is satisfied, if and only if, $x = x_0$, $y = y_0$.

This is specified by saying

$$\text{(C)} \qquad \begin{aligned} ax_0 + by_0 &= e \\ cx_0 + dy_0 &= f \end{aligned}$$

are true statements, with $D \neq 0$.

From our definition,

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix},$$

and from Theorem I, for $x = x_0$, we may write

$$x_0 D = \begin{vmatrix} ax_0 & b \\ cx_0 & d \end{vmatrix},$$

and from Theorem II, for $y = y_0$, $x = x_0$,

$$x_0 D = \begin{vmatrix} ax_0 + by_0 & b \\ cx_0 + dy_0 & d \end{vmatrix}.$$

But from (C) this may be rewritten

$$x_0 D = \begin{vmatrix} e & b \\ f & d \end{vmatrix}.$$

Thus,

$$x_0 = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{D} = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

$$y_0 = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{D} = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

which is Cramer's Rule.

We see this is true provided $D \neq 0$.
Determinant $D = 0$ if the two linear equations are inconsistent, (Graphs are distinct parallel lines.) or redundant (Graphs are the same line.). So that if the graphs (lines) are not parallel or coincident, then the common point of in-

tersection has the unique values of x_0 and y_0 as given by Cramer's Rule.

6. A CLEVER DEVICE IN ACHIEVING AN INDUCTIVE TRANSITION

From Definition (A) $F_1 = 1$, $F_2 = 1$ and

$$F_{n+2} = F_{n+1} + F_n$$

Suppose we write two examples of this (for $n = k$ and $n = k-1$).

$$F_{k+2} = F_{k+1} + F_k$$

$$F_{k+1} = F_k + F_{k-1}$$

Let us try to solve the pair of simultaneous linear equations.

$$\text{(D)} \qquad \begin{aligned} F_{k+2} &= x\,F_{k+1} + y\,F_k \\ F_{k+1} &= x\,F_k + y\,F_{k-1}\,. \end{aligned}$$

This is silly because we know the answer is $x_0 = 1$ and $y_0 = 1$, but using Cramer's Rule we note:

$$\text{(E)} \qquad y_0 = 1 = \frac{\begin{vmatrix} F_{k+1} & F_{k+2} \\ F_k & F_{k+1} \end{vmatrix}}{\begin{vmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{vmatrix}} = \frac{F_{k+1}^2 - F_k F_{k+2}}{F_{k+1}F_{k-1} - F_k^2}$$

Let us now use Mathematical Induction to prove identity III which is

$$P(n): \qquad F_{n+1}\,F_{n-1} - F_n^2 = (-1)^n$$

If we note that $F_0 = 0$ is valid, then

$$P(1): \quad F_2 F_0 - F_1^2 = (-1)^1 = -1$$

is true. Thus part A is done.

Suppose P(k) is true. From (E),

$$1 = \frac{F_{k+1}^2 - F_k F_{k+2}}{F_{k+1} F_{k-1} - F_k^2}$$

so that

$$P(k+1): \quad F_{k+2} F_k - F_{k+1}^2 = (-1)^{k+1}$$

is indeed true!! Thus part B is done.

The proof is complete by mathematical induction and part C is done.

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SPECIAL NOTICE

The Fibonacci Association has on hand 22 copies of Dov Jarden, Recurrent Sequences, Riveon Lematematika, Jerusalem, Israel. This is a collection of papers on Fibonacci and Lucas numbers with extensive tables of factors extending to the 385th Fibonacci and Lucas numbers. The volume sells for $5.00 and is an excellent investment. Check or money order should be sent to Verner Hoggatt at San Jose State College, San Jose, Calif.

REQUEST

Maxey Brooke would like any references suitable for a Lucas bibliography. His address is 912 Old Ocean Ave., Sweeny, Tex.