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DIVISIBILITY I

Many of the most interesting properties of the famous Fibonacci numbers ($F_1 = F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 8$, ..., $F_{i+1} = F_i + F_{i-1}$, ...) depend on the notion of divisibility.

1. Definitions. We say that the Fibonacci number 8 is exactly divisible by 4 since 8 = 4x2. (Usually the word "exactly" is omitted.) Note that the following statements are equivalent:

(i) 8 is divisible by 4
(ii) 8 is a multiple of 4
(iii) 4 is a factor of 8
(iv) 4 is a divisor of 8
(v) 4 divides 8 (abbreviated: $4 \mid 8$).

Recalling that an integer is merely a whole number (0 included!), we may now say that a positive integer greater than one is a prime if and only if it has exactly two divisors. Thus F_3, F_4, and F_5 are primes, but $F_6 = 8$ is composite since it has four factors: 1, 2, 4, 8. Note that 2 is the smallest prime--and the only even one. In the interest of concise statements, 1 is normally not considered a prime (although it shares the property of "being divisible only by 1 and itself"). I.e., if we called 1 a prime, many theorems in elementary number theory would have to be reworded.

Digressing a moment we might observe that the following is an unsolved problem.

> Is there a composite Fibonacci number with exactly three divisors?

If there were, then it would have to be the square of a prime--but 1 and 144 are the only known square Fibonacci numbers. In Tomorrow's Math, C.S. Ogilvy reports that one investigator is close to a solution of this problem.

2. Tests for Divisibility. In order to prove that, say, the Fibonacci number 987 is composite, it suffices to find a single divisor n such that $1 < n < 987$. Now certainly 987 is not divisible by 2 since it doesn't end in an even digit (0, 2, 4, 6, 8). But we see that 3 divides 987. Perhaps you were able to reach this conclusion mentally by noting that $9 + 8 + 7 = 24$ = a multiple of 3. This procedure, commonly called casting out 3's, is one of several simple tests for divisibility. For convenience we list some of the simpler tests for divisibility.

N is divisible by 2 if and only if it ends in an even digit.

N is divisible by 3 if and only if the sum of the digits of N is a multiple of 3.

N is divisible by 4 if and only if the number consisting of the last two digits of N is a multiple of 4.

N is divisible by 5 if and only if N ends in "0" or "5".

N is divisible by 6 if and only if N is a multiple of both 2 and 3.

N is divisible by 8 if and only if the number consisting of the last three digits of N is a multiple of 8.

N is divisible by 9 if and only if the sum of the digits of N is a multiple of 9.

Many other tests are easy to formulate: N is a multiple of 10 if and only if N ends in "0"; N is a multiple of 12 if and only if both 3 and 4 are factors of N. Some caution, however, is desirable. A number can be divisible by both 3 and 6, and yet not be divisible by 18. (Find an example.)

3. Prime or Composite? Suppose that you are confronted with the problem of determining whether or not $F_{13} = 233$ is a prime. How much work is involved? Assume that you may use any of the previously mentioned tests "free of charge". How many additional questions of the form "Is 233 divisible by so-and-so?", must you ask?

Of course there is no need to check for divisibility by 6 (or any other composite number) for if 233 is not itself a prime, it will have to have a prime factor less than 233. Fortunately we need not try each of the 50 primes less than 233. The simple but remarkable fact is that we can get by with no more than three "questions" (assuming that we test for divisibility by 2, 3, and 5 mentally): Is 233 divisible by 7? 11? 13?

In the case of 233, it turns out that the answer to each of these questions is "no". Let us see why this means that 233 must be a prime. If 233 were composite, it would have to have a prime factor $p > 13$. Thus $q = 233/p$ would be an integer, greater than 1 but less than 17; i.e., q, and hence 233, would have to be divisible by at least one of 2, 3, 5, 7, 11 or 13--a contradiction.

4. Problems. Solutions to these problems may be found on page 64.

1.1. What do you notice about every third Fibonacci number? Every fourth? Every fifth?

1.2. Try to guess a generalization of problem 1.1.

1.3. It is desirable to be able to define the greatest common divisor d of two integers (not both zero) without using the word "greatest". Do this, using only the following words and symbols: d, a, b, k, $|$, and, if, then.

1.4. Using 987 as an example, explain why the test for divisibility by 9 works.

1.5. What is the minimum number of questions that you need to ask in order to determine whether or not $F_{19} = 4181$ is a prime?

1.6. Let S be the following set of numbers:

4	7	10	13	16	19	22	...
7	12	17	22	27	32	37	...
10	17	24	31	38	45	52	...
13	22	31	40	49	58	67	...
16	27	38	49	60	71	82	...
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.	.	.	.	.	.	.	

Prove that (a) if N is in S, then 2N + 1 is composite and (b) if N is not in S, then 2N + 1 is a prime. Assume that the numbers in each row form an arithmetic progression and that the first column is the same as the first row.

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