EXPLORING RECURRENT SEQUENCES

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The following article constitutes the Elementary Research Department of the present issue of the Fibonacci Quarterly. Readers are requested to send their discoveries, queries, and suggestions dealing with this portion of the Quarterly to Brother U. Alfred, St. Mary's College, Calif.

Everyone who buys insurance is urged to read the fine print because it usually contains qualifications of an important nature. In a similar vein the readers of the newly created Fibonacci Quarterly should turn to the inside cover and examine the sub-title: "A journal devoted to the study of integers with special properties." This in no way indicates that the editors could not fill the pages of their magazine with material dealing exclusively with Fibonacci sequences. It does, however, provide for a measure of latitude and a certain variety in the contents while adhering to the main theme indicated by the title of the magazine. In this spirit, the "Fibonacci explorers" are invited to look into a somewhat broader topic: Recurrent Sequences.

The word "recurrent" need not frighten anyone. Recurrence simply means repetition. A sequence is a set of mathematical quantities that are ordered in the same way as the integers: 1, 2, 3,⋯. Put the two ideas together and the result is a "recurrent sequence."

Perhaps the simplest example of such a sequence is the integers themselves. Let us denote the terms of our sequence by T_1, T_2, $T_3, \cdots, T_n \cdots$. In the case of the integers, the relation involved is

$$T_{n+1} = T_n + 1$$

that is, every integer is one more than the integer preceding it. This idea is readily extended to even integers and odd integers. If, for example, T_n is an even integer the next even integer is

$$T_{n+1} = T_n + 2 \quad .$$

Likewise, if T_n is an odd integer, the next odd integer is

$$T_{n+1} = T_n + 2 \quad .$$

Now look at the last two laws of recurrence. They are the same! This fact is a source of confusion to students of elementary algebra who think that if x and $x + 2$ represent consecutive even integers, something else would represent consecutive odd integers. The answer lies, of course, in the "if" portion of the proposition. If x is an odd integer, then $x + 2$ is also the next odd integer.

The natural extension of such relations which we have been considering is the arithmetic progression in which each term differs from the preceding term by a fixed quantity, a, called the common difference. Thus for this type of sequence we have

$$T_{n+1} = T_n + a \; ,$$

where a can be any real or complex quantity.

Another well-known type of recurrence sequence is the geometric progression in which each term is a fixed multiple, r, of the previous term. The relation in this case is

$$T_{n+1} = r\,T_n \; .$$

A simple example is: 2, 6, 18, 54, 162, ···, where $r = 3$, $T_1 = 2$.

We now come to the commercial. Recurrent sequences in which each term is the sum of the two preceding terms are known as Fibonacci sequences. The law of recurrence for all such sequences is

$$T_{n+1} = T_n + T_{n-1} \; .$$

Starting with the values of T_1 and T_2, it is possible to build up such a sequence. Thus, if $T_1 = 3$ and $T_2 = 11$, it follows that $T_3 = 14$, $T_4 = 25$, $T_5 = 39$, ··· .

One can go on to variations of this idea. For example:

$$T_{n+1} = 2\,T_n + 3\,T_{n-1}$$

or

$$T_{n+1} = T_n + T_{n-1} + T_{n-2}$$

Any one such sequence can be the subject of a great deal of investigation and research which can lead to many interesting mathematical results.

At this juncture it may be well to point out that in some instances, the law of recurrence is such that it is possible to work out an explicit mathematical expression for the nth term. In others, this is not convenient or possible. For example; if $T_1 = 1$, $T_2 = 1$, and every term is the sum of all the terms preceding it, we find directly that $T_3 = 2$, $T_4 = 4$, $T_5 = 8$, $T_6 = 16$, $T_7 = 32, \cdots$, so that a person not endowed with mathematical genius can see that the nth term is given by ...... ?

On the other hand, if $T_1 = 1$, $T_2 = 1$ and

$$T_{n+1} = T_n + T_{n-1} \quad ,$$

we have the well-known Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, ⋯ whose terms are such that they are not as readily expressible by a simple formula. Hence, we establish them as a standard sequence which can serve to express results found in other sequences.

EXPLORATION

A few sequences worthy of exploration have already been indicated. Other suggestions follow, and beginning readers are urged to create additional sequences of their own. Interesting mathematical results derived from such work should be communicated to the Editor of this department of the Fibonacci Quarterly. Here are a few suggestions to start you exploring:

1. Let $T_1 = a$, $T_2 = b$, where a and b are any positive numbers, and let the law of formation in the sequence be that each term is the quotient of the two preceding terms.
2. Starting with the same initial terms, let each term be the product of the two previous terms.
3. Another law: Let each odd-numbered term be the sum of the two previous terms and each even-numbered term be the difference of the two previous terms.
4. Let each odd-numbered term be the product of the two preceding terms and each even-numbered term be the quotient of the two preceding terms.
5. Starting with $T_1 = a$, $T_2 = b$, $T_3 = c$, let the law of formation be:

$$T_{n+1} = T_n + T_{n-1} - T_{n-2} \quad .$$