A PRIMER ON THE FIBONACCI SEQUENCE – PART II

S. L. BASIN AND VERNER E. HOGGATT, JR., SAN JOSE STATE COLLEGE

1. INTRODUCTION

The proofs of existing Fibonacci identities and the discovery of new identities can be greatly simplified if matrix algebra and a particular 2 x 2 matrix are introduced. The matrix approach to the study of recurring sequences has been used for some time [1] and the Q matrix appeared in a thesis by C. H. King [2]. We first present the basic tools of matrix algebra.

THE ALGEBRA OF (TWO-BY-TWO) MATRICES

The two-by-two matrix A is an array of four elements a, b, c, d:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The zero matrix, Z, is defined as,

$$Z = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

The identity matrix, I, is

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The matrix C, which is the <u>matrix sum</u> of two matrices A and B, is

$$C = A + B = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}.$$

The matrix P, which is the <u>matrix product</u> of two matrices A and B, is

$$P = AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}.$$

The <u>determinant D(A)</u> of matrix A is

$$D(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Two matrices are equal if and only if the corresponding elements are equal; that is,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} = B,$$

if and only if, $a = e$, $b = f$, $c = g$, $d = h$.

A SIMPLE THEOREM

The determinant, $D(P)$, of the product, $P = AB$, of two matrices A and B is the product of the determinants $D(A)$ and $D(B)$

$$D(P) = D(AB) = D(A)\,D(B)$$

The proof is left as a simple exercise in algebra.

THE Q MATRIX

The Q matrix and the determinant of Q, $D(Q)$, are:

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad D(Q) = -1 .$$

If we designate $Q^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$, then

$$Q = Q^1 = Q^0 Q = IQ = QI = QQ^0 .$$

DEFINITION: $Q^{n+1} = Q^n Q^1$, an inductive definition where $Q^1 = Q$. This is the law of exponents for matrices.

It is easily proved by mathematical induction that

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} ,$$

where F_n is the nth Fibonacci number, and the determinant of Q^n is

$$D(Q^n) = D^n(Q) = (-1)^n .$$

MORE PROOFS

We may now prove several of the identities very nicely. Let us prove identity III (given in Part I), that is,

$$F_{n+1} F_{n-1} - F_n^2 = (-1)^n .$$

Proof:

$$\text{If} \quad Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \text{ and } D(Q^n) = (-1)^n ,$$

then

$$D(Q^n) = \begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = F_{n+1} F_{n-1} - F_n^2 - (-1)^n$$

Let us prove identity VI

$$F_{2n+1} = F_{n+1}^2 + F_n^2 .$$

since

$$Q^{n+1} Q^n = Q^{2n+1},$$

then

$$Q^n Q^{n+1} = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \begin{pmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{pmatrix}$$

$$= \begin{pmatrix} F_{n+1} F_{n+2} + F_n F_{n+1} & F^2_{n+1} + F^2_n \\ F_n F_{n+2} + F_{n-1} F_{n+1} & F_n F_{n+1} + F_{n-1} F_n \end{pmatrix}$$

But this is also

$$Q^{2n+1} = \begin{pmatrix} F_{2n+2} & F_{2n+1} \\ F_{2n+1} & F_{2n} \end{pmatrix}$$

Since these two matrices are equal we may equate corresponding elements so that

$$F_{2n+2} = F_{n+1} F_{n+2} + F_n F_{n+1} \quad \text{(Upper Left)}$$

$$F_{2n+1} = F^2_{n+1} + F^2_n \quad \text{(Upper Right)}$$

$$F_{2n+1} = F_n F_{n+2} + F_{n-1} F_{n+1} \quad \text{(Lower Left)}$$

$$F_{2n} = F_n F_{n+1} + F_{n-1} F_n \quad \text{(Lower Right)}$$

$$= F_n (F_{n+1} + F_{n-1})$$

If we accept identity V: $L_n = F_{n+1} + F_{n-1}$, then

$$F_{2n} = F_n L_n$$

which gives identity VIII. Return again to

$$F_{2n} = F_n (F_{n+1} + F_{n-1})$$

From $F_{k+2} = F_{k+1} + F_k$, for $k = n - 1$, one can write $F_n = F_{n+1} - F_{n-1}$, thus also

$$F_{2n} = (F_{n+1} - F_{n-1})(F_{n+1} + F_{n-1}) = F_{n+1}^2 - F_n^2$$

which is identity VII.

It is a simple task to verify

$$Q^2 = Q + I$$

and

$$Q^{n+2} = Q^{n+1} + Q^n ,$$

and

$$Q^n = Q F_n + I F_{n-1} ,$$

where F_n is the nth Fibonacci number and the multiplication of matrix A, by a number q, is defined by

$$qA = q\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} aq & bq \\ cq & dq \end{pmatrix} .$$

GENERATION OF FIBONACCI NUMBERS BY LONG DIVISION

$$\frac{1}{1 - x - x^2} = F_1 + F_2 x + F_3 x^2 + \cdots + F_n x^{n-1} + \ldots$$

In the process of long division below

$$1 - x - x^2 \,\overline{\big)\; 1 \quad}$$

there is no ending. As far as you care to go the process will yield Fibonacci Numbers as the coefficients.

F_n AS A FUNCTION OF ITS SUBSCRIPT

It is not difficult to show by mathematical induction that

$$P(n): \qquad F_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$$

This can be derived in many ways. P(1) and P(2) are clearly true. From $F_k = F_{k-1} + F_{k-2}$ and the inductive assumption that P(k-2) and P(k-1) are true, then

$$\text{(a)} \qquad F_{k-2} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k-2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-2}\right\}$$

$$\text{(b)} \qquad F_{k-1} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right\}$$

Adding, after a simple algebra step, we get

$$F_{k-1} + F_{k-2} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k-2}\left(\frac{1+\sqrt{5}}{2} + 1\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k-2}\left(\frac{1-\sqrt{5}}{2} + 1\right)\right\}$$

Observing that

$$\frac{1+\sqrt{5}}{2} + 1 = \frac{3+\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^2$$

$$\frac{1-\sqrt{5}}{2} + 1 = \frac{3-\sqrt{5}}{2} = \left(\frac{1-\sqrt{5}}{2}\right)^2$$

it follows simply that if (a) and (b) are true (P(k - 2) and P(k - 1) are true), then for n = k,

$$P(k): \quad F_k = F_{k-1} + F_{k-2} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right\}$$

The proof is complete by mathematical induction. Similarly it may be shown that

$$L_n = F_{n+1} + F_{n-1}$$

and

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n .$$

Let us now prove identity VIII

$$F_{2n} = F_n L_n$$

Proof:

$$F_{2n} = \frac{1}{\sqrt{5}}\left\{\left[\left(\frac{1+\sqrt{5}}{2}\right)^n\right]^2 - \left[\left(\frac{1-\sqrt{5}}{2}\right)^n\right]^2\right\}$$

Now factoring:

$$F_{2n} = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$$

$$F_{2n} = F_n L_n .$$

MORE IDENTITIES

XIV. $F_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$

XV. $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$

XVI. $F_1^3 + F_2^3 + \cdots + F_n^3 = \frac{F_{3n+2} + (-1)^{n+1}\, 6 F_{n-1} + 5}{10}$

XVII. $1 \cdot F_1 + 2F_2 + 3F_3 + \cdots + nF_n = (n+1)F_{n+2} - F_{n+4} + 2$

XVIII. $F_2 + F_4 + \qquad + F_{2n} = F_{2n+1} - 1$

XIX. $F_1F_2 + F_2F_3 + F_3F_4 + \cdots + F_{n-1}F_n = \frac{1}{2}(F_{n+2} - F_nF_{n-1}) - 1$

XX. $\sum_{i=0}^{n} \binom{n}{i} F_{n-1} = F_{2n}$,

where $\binom{n}{i} = \frac{n!}{(n-i)!\,i!}$ and $m! = 1 \cdot 2 \cdot 3 \cdot \cdots m$.

XXI. $F_{3n+3} = F_{n+1}^3 + F_{n+2}^3 - F_n^3$

XXII. $F_nF_m - F_{n-k}F_{m+k} = (-1)^{n-k}F_kF_{m+k-n}$

REFERENCES

1. J. S. Frame, "Continued fractions and matrices," Amer. Math. Monthly, Feb. 1949, p. 38.
2. Charles H. King, 'Some properties of the Fibonacci numbers,' Master's Thesis, San Jose State College, June, 1960.

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