FIBONACCI MATRICES AND LAMBDA FUNCTIONS

MARJORIE BICKNELL AND VERNER E. HOGGATT, JR., SAN JOSE STATE COLLEGE, SAN JOSE, CALIF.

When we speak of a Fibonacci matrix, we shall have in mind matrices which contain members of the Fibonacci sequence as elements. An example of a Fibonacci matrix is the Q matrix as defined by King in [1], pp. 11-27, where

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} .$$

The determinant of Q is -1, written $\det Q = -1$. From a theorem in matrix theory,

$$\det Q^n = (\det Q)^n = (-1)^n .$$

By mathematical induction, it can be shown that

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} ,$$

so that we have the familiar Fibonacci identity

$$F_{n+1} F_{n-1} - F_n^2 = (-1)^n .$$

The lambda function of a matrix was studied extensively in [2] by Fenton S. Stancliff, who was a professional musician. Stancliff defined the lambda function $\lambda(M)$ of a matrix M as the change in the value of the determinant of M when the number one is added to each element of M. If we define $(M + k)$ to be that matrix formed from M by adding any given number k to each element of M, we have the identity

(1) $$\det (M + k) = \det M + k\,\lambda(M).$$

For an example, the determinant $\lambda(Q^n)$ is given by

$$\lambda(Q^n) = \begin{vmatrix} F_{n+1} + 1 & F_n + 1 \\ F_n + 1 & F_{n-1} + 1 \end{vmatrix} - \det Q^n$$

$$= (F_{n+1}F_{n-1} - F_n^2) + (F_{n-1} + F_{n+1} - 2F_n) - \det Q^n$$

$$= F_{n-3}$$

which follows by use of Fibonacci identities. Now if we add k to each element of Q^n, the resulting determinant is

$$\begin{vmatrix} F_{n+1} + k & F_n + k \\ F_n + k & F_{n-1} + k \end{vmatrix} = \det Q^n + k F_{n-3}\,.$$

However, there are more convenient ways to evaluate the lambda function. For simplicity, we consider only 3 x 3 matrices.

THEOREM. For the given general 3 x 3 matrix M, $\lambda(M)$ is expressed by either of the expressions (2) or (3). For

$$M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix} ,$$

(2) $$\lambda(M) = \begin{vmatrix} a + e - (b + d) & b + f - (c + e) \\ d + h - (g + e) & e + j - (h + f) \end{vmatrix} ;$$

or

(3) $$\lambda(M) = \begin{vmatrix} 1 & b & c \\ 1 & e & f \\ 1 & h & j \end{vmatrix} + \begin{vmatrix} a & 1 & c \\ d & 1 & f \\ g & 1 & j \end{vmatrix} + \begin{vmatrix} a & b & 1 \\ d & e & 1 \\ g & h & 1 \end{vmatrix} .$$

Proof: This is made by direct evaluation and a simple exercise in algebra.

An application of the lambda function is in the evaluation of determinants. Whenever there is an obvious value of k such that det (M + k) is easy to evaluate, we can use equation (1) advantageously. To illustrate this fact, consider the matrix

$$M = \begin{pmatrix} 1000 & 998 & 554 \\ 990 & 988 & 554 \\ 675 & 553 & 554 \end{pmatrix} \quad .$$

We notice that, if we add $k = -554$ to each element of M, then $\det(M+k) = 0$ since every element in the third column will be zero. From (2) we compute

$$\lambda(M) = \begin{vmatrix} 0 & 10 \\ -120 & 435 \end{vmatrix} = 1200 ;$$

and from (1) we find that

$$0 = \det M + (-554)(1200) ,$$

so that $\det M = (554)(1200)$.

Readers who enjoy mathematical curiosities can create determinants which are not changed in value when any given number k is added to each element, by writing any matrix D such that $\lambda(D) = 0$.

LEMMA: If two rows (or columns) of a matrix D have a constant difference between corresponding elements, then $\lambda(D) = 0$.

Proof: Evaluate $\lambda(D)$ directly, by (2) or (3).

For example, we write the matrix D, where corresponding elements in the first and second rows differ by 4, such that

$$\det D = \begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 4 & 9 & 8 \end{vmatrix} = \begin{vmatrix} 1+k & 2+k & 3+k \\ 5+k & 6+k & 7+k \\ 4+k & 9+k & 8+k \end{vmatrix} = 24$$

Now, we consider other Fibonacci matrices. Suppose that we want to write a Fibonacci matrix U such that $\det U = F_n$. Now

$$\begin{vmatrix} a & 0 & 0 \\ x & b & 0 \\ y & z & d \end{vmatrix} = abd.$$

We can write $F_n = F_1F_1F_n = F_1F_2F_n = F_2F_2F_n$ for any n, and for some n we will also have other Fibonacci factorizations. Hence, $F_n = \det U$ for

$$U = \begin{pmatrix} F_1 & F_0 & F_0 \\ F_m & F_2 & F_0 \\ F_k & F_p & F_n \end{pmatrix}$$

where $F_0 = 0$. If we choose $m = k = 3$ and $p = 2$, we find that $\lambda(U) = 0$. If we choose $m = 1$ or 2, $k = 1$ or 2, and let p be an arbitrary integer, then $\lambda(U) = F_n$.

A more elegant way to write such a matrix was suggested by Ginsburg in [3], who showed that if $a = F_{2p}$, $c = b = F_{2p+1}$, $d = e = F_{2p+2}$, and $f = F_{2p+3}$, then $\det B = n$, where

$$B = \begin{pmatrix} a & b & n \\ c & d & n \\ e & f & n \end{pmatrix}.$$

Letting $n = F_m$, we can write $F_m = \det U$, where

$$U = \begin{pmatrix} F_{2p} & F_{2p+1} & F_m \\ F_{2p+1} & F_{2p+2} & F_m \\ F_{2p+2} & F_{2p+3} & F_m \end{pmatrix}.$$

Using equation (3) we have

$$\lambda(U) = \begin{vmatrix} 1 & b & F_m \\ 1 & d & F_m \\ 1 & f & F_m \end{vmatrix} + \begin{vmatrix} a & 1 & F_m \\ c & 1 & F_m \\ e & 1 & F_m \end{vmatrix} + \begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$$

$$= 0 + 0 + 1/F_m (\det U) = 1$$

If we let $k = F_{m-1}$, from (1) we see that

$$\det(U + F_{m-1}) = F_m + (F_{m-1})(1) = F_{m+1} \,.$$

Notice the possibilities for finding Fibonacci identities using the lambda function and evaluation of determinants. As a brief example, we let $k = F_n$ and consider det $(Q^n + F_n)$, which gives us

$$\begin{vmatrix} F_{n+1} + F_n & F_n + F_n \\ F_n + F_n & F_{n-1} + F_n \end{vmatrix} = \det Q^n + F_n \lambda(Q^n)$$

or

$$\begin{vmatrix} F_{n+2} & 2F_n \\ 2F_n & F_{n+1} \end{vmatrix} = (-1)^n + F_n F_{n-3}$$

so that

$$4F_n^2 = F_{n+2}F_{n+1} - F_n F_{n-3} + (-1)^{n+1} \,.$$

As a final example of a Fibonacci matrix, we take the matrix R, given by

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix} .$$

which has been considered by Brennan [4].

It can be shown by mathematical induction that

$$R^n = \begin{pmatrix} F_{n-1}^2 & F_{n-1}F_n & F_n^2 \\ 2F_{n-1}F_n & F_{n+1}^2 - F_{n-1}F_n & 2F_nF_{n+1} \\ F_n^2 & F_nF_{n+1} & F_{n+1}^2 \end{pmatrix}$$

The reader may verify by equation (2) and by Fibonacci identities that

$$\lambda(R^n) = \begin{vmatrix} F^2_{n-1} + F^2_{n+1} - 4F_{n-1}F_n & 2F_{n-1}F_n + 2F_nF_{n+1} - F^2_n - F^2_{n+1} \\ 3F_{n-1}F_n - F^2_n - F^2_{n+1} + F_nF_{n+1} & 2F^2_{n+1} - 3F_nF_{n+1} - F_nF_{n-1} \end{vmatrix}$$

$$= \begin{vmatrix} F_{2n-3} & F_{2n-2} \\ -F^2_{n-2} & -F_{n-2}F_{n-1} + (-1)^n \end{vmatrix} = (-1)^n (F^2_{n-1} - F_{n-3}F_{n-2}) .$$

Here we see that the value of (R^n) is the center element of R^{n-2} multiplied by $(-1)^n$.

REFERENCES

1. Charles H. King, Some Properties of the Fibonacci Numbers, (Master's Thesis) San Jose State College, June, 1960.
2. From the unpublished notes of Fenton S. Stancliff.
3. Jukethiel Ginsburg, "Determinants of a Given Value," Scripta Mathematica, Vol. 18, issues 3-4, Sept.-Dec., 1952, p. 219.
4. From the unpublished notes of Terry Brennan.