LINEAR RECURRENCE RELATIONS – PART I

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1. INTRODUCTION

Most of the special sequences, which appear in The Fibonacci Quarterly, satisfy a type of equation called a recurrence relation i. e., a difference equation whose independent variable is restricted to integral values. Although there arc several good textbooks (e.g., see [1], [2], [3] or [4]) which present various methods of solution for many such equations, the beginner may not be acquainted with any of them, and in fact is likely to have more knowledge of the theory of differential equations than of that of recurrence relations.

The purpose of this series of articles is to introduce the beginner to the subject, and to derive explicit expressions for the solution of certain general, linear recurrence relations by applying a generating function transformation. The particular generating function chosen is seldom used in the treatment of recurrence relations, but for the purpose of developing general formulas it has the advantage of immediately transforming the problem to a more familiar one involving differential equations, for which there is already available a great wealth of special formulas and techniques.

2. DEFINITIONS

A linear recurrence relation of order k is an equation of the form

$$\sum_{j=0}^{k} a_{j,n}\, y_{n+j} = b_n \quad , \tag{2.1}$$

where $a_{0,n}, a_{1,n}, \cdots, a_{k,n}$ and b_n are given functions of the independent variable n over the set of consecutive non-negative integers S, and $a_{0,n}a_{k,n} \neq 0$ on S. If $b_n \equiv 0$, the relation is called homogeneous, otherwise it is said to be non-homogeneous. We may introduce the translation operator E^m, defined by

$$E^m y_n = y_{n+m} \qquad (m = 0, 1, \cdots, k) \quad , \tag{2.2}$$

and thus we can write (2.1) as

$$L_k(E)\, y_n = b_n \quad , \tag{2.3}$$

where the linear operator $L_k(E)$ is

$$L_k(E) = \sum_{j=0}^{k} a_{j,n} E^j \tag{2.4}$$

A sequence whose terms are v_n is a solution to the recurrence relation on the set S if the substitution $y_n = v_n$ reduces relation (2.3) to an identity on S.

If a set of k successive initial values $y_0, y_1, \cdots, y_{k-1}$ is given arbitrarily, equation (2.1) or (2.3) enables us to extend this set to $k+1$ successive values. Using mathematical induction, it can be easily established that the recurrence relation (2.3) over the set S of consecutive non-negative integers has one and only one solution for which the k values are prescribed.

3. A SERIES TRANSFORM

For the sequence $\{y_n\}$, $n = 0, 1, 2, \cdots$, we introduce the exponential generating function defined by tne infinite series

$$Y(t) = \sum_{n=0}^{\infty} y_n \frac{t^n}{n!}, \tag{3.1}$$

which we suppose is convergent for some positive value of t. From (3.1) we find the derived series

$$\frac{d^jY}{dt^j} \equiv Y^{(j)}(t) = \sum_{n=0}^{\infty} y_{n+j} \frac{t^n}{n!} \qquad (j = 0, 1, \cdots, k). \tag{3.2}$$

These series of course have the same radius of convergence as (3.1), and are seen as the generating functions of the sequences $\{y_{n+j}\}$, $j = 0, 1, \cdots, k$. Now from (3.1), we have the inverse transform

$$y_n = Y^{(n)}(0) = \left[\frac{d^n}{dt^n} Y(t)\right]_{t=0} \qquad (n = 0, 1, 2, \cdots, k). \tag{3.3}$$

The relations (3.2) and (3.3) follow from known properties of Taylor series.

4. EXPLICIT SOLUTION OF A LINEAR RECURRENCE RELATION

We shall now derive the formula for the general solution to the linear homogeneous recurrence relation

$$\sum_{j=0}^{k} a_j y_{n+j} \equiv L_k(E) y_n = 0 \tag{4.1}$$

with constant coefficients. (Discussion of the non-homogeneous case will appear in the next issue of this journal.) The derivation is based on the application of the exponential generating function (3.1) which transforms the recurrence relation into a more familiar differential equation.

Multiplying both sides of (4.1) by $t^n/n!$ and summing over n from 0 to ∞, we thus obtain the transformed equation

$$\sum_{j=0}^{k} a_j Y^{(j)}(t) = L_k(D) Y = 0 , \quad \left(D \equiv \frac{d}{dt}\right) , \tag{4.2}$$

which is an ordinary linear differential equation of order k. Now it is well known[1] that, if $r_1, r_2, \cdots, r_k$ are k distinct roots of the characteristic equation

$$L_k(r) = 0, \tag{4.3}$$

then the general solution of (4.2) is given by

$$Y(t) = \sum_{i=1}^{k} c_i e^{r_i t} , \tag{4.4}$$

where c_i are k arbitrary constants. Application of the inverse transform (3. then yields immediately the explicit formula

[1]See for example, almost any textbook on ordinary differential equations.

$$y_n = \sum_{i=1}^{k} c_i r_i^n \tag{4.5}$$

for the general solution of the recurrence relation (4.1).

In the case where the characteristic equations $L_k(r) = 0$ possesses m distinct roots $r_1, r_2, \cdots, r_m$ and each root r_i being of multiplicity m_i $(i = 1, \cdots, m)$, with

$$\sum_{i=1}^{m} m_i = k , \tag{4.6}$$

the differential equation (4.2) is known to have the general solution

$$Y(t) = \sum_{i=1}^{m} e^{r_i t} \sum_{j=0}^{m_i - 1} b_{ij} t^j , \tag{4.7}$$

where b_{ij} are k arbitrary constants. Applying the inverse transform (3.3), we then obtain the general solution

$$y_n = \sum_{i=1}^{m} r_i^n \sum_{j=0}^{m_i - 1} b_{ij} n^j \tag{4.8}$$

to the recurrence relation (4.1).

In Part II of this article, we shall not only derive an explicit formula for the general solution of the non-homogeneous linear recurrence relation with constant coefficients, but shall also show how the method employing the exponential generating function may solve certain recurrence relations having variable coefficients.

5 EXAMPLE

The Fibonacci numbers satisfy the second-order recurrence relation

$$F_{n+2} - F_{n+1} - F_n = 0, \quad F_0 = 0, \quad F_1 = 1 \tag{5.1}$$

or

$$(5.2) \qquad L_2(E)\,F_n = 0 \quad ,$$

where

$$(5.3) \qquad L_2(E) = E^2 - E - 1.$$

The characteristic equation $L_2(r) = 0$ has the distinct roots

$$(5.4) \qquad r_1 = (1 + \sqrt{5})/2, \qquad R_2 = (1 - \sqrt{5})/2 \quad ,$$

so that the formula (4.5) immediately yields

$$(5.5) \qquad F_n \equiv y_n = c_1 r_1^n + c_2 r_2^n \quad .$$

Now since $F_0 = 0$, $F_1 = 1$, we obtain $c_1 = -c_2 = 1/\sqrt{5}$; hence the general solution for the Fibonacci sequence is expressed by

$$(5.6) \qquad F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1 + \sqrt{5}}{2}\right)^n - \left(\frac{1 - \sqrt{5}}{2}\right)^n\right] \quad .$$

We note from (4.2) that the transformed equation for (5.1) is the second-order differential equation

$$(5.7) \qquad Y'' - Y' - Y = 0, \quad Y(0) = 0, \quad Y'(0) = 1 \quad .$$

Hence the exponential generating function for the Fibonacci sequence is

$$(5.8) \qquad Y(t) = \left[e^{r_1 t} - e^{r_2 t}\right]\Big/\sqrt{5} = \sum_{n=0}^{\infty} F_n \frac{t^n}{n!} \quad ,$$

while the well-known ordinary generating function for this sequence is

$$W(t) = \frac{t}{1 - t - t^2} = \sum_{n=0}^{\infty} F_n t^n \quad . \tag{5.9}$$

The two generating functions $W(t)$ and $Y(t)$ are related by the expression

$$W(t) = \int_0^{\infty} e^{-z}\, Y(tz)\, dz \quad . \tag{5.10}$$

REFERENCES

1. L. M. Milne-Thompson, The Calculus of Finite Differences, London, 1933.
2. C. Jordan, Calculus of Finite Differences, New York, 2nd Ed., 1947.
3. S. Goldberg, Introduction to Difference Equations, New York, 1958.
4. G. Boole, Calculus of Finite Differences, New York, 4th Ed., 1926.

PROBLEM DEPARTMENT

P-1. The recurrence relation for the sequence of Lucas numbers is

$$L_{n+2} - L_{n+1} - L_n = 0 \text{ with } L_1 = 1, \; L_2 = 3 \; .$$

Find the transformed equation, the exponential generating function, and the general solution.

P-2. Find the general solution and the exponential generating function for the recurrence relation

$$y_{n+3} - 5y_{n+2} + 8y_{n+1} - 4y_n = 0 \; ,$$

with $y_0 = 0, \; y_1 = 0, \; y_2 = -1.$

REQUEST

Maxey Brooke would like any references suitable for a Lucas bibliography. His address is 912 Old Ocean Ave., Sweeny, Tex.