SOME FIBONACCI RESULTS USING FIBONACCI-TYPE SEQUENCES

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The elements of the Fibonacci sequence satisfy the recursion formula, $F_{n+1} = F_n + F_{n-1}$, where $F_0 = 0$ and $F_1 = 1$. Let us define an F-sequence as one for which the recursion formula $u_{n+1} = u_n + u_{n-1}$ holds for the elements u_n of the sequence.

Suppose $\{u_n\}$ and $\{v_n\}$ are two F-sequences. Then a linear combination, $\{cu_n + dv_n\}$, is also an F-sequence. If the determinant

$$\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \neq 0 ,$$

then by an application of a theorem from algebra, every F-sequence can be expressed as a unique linear combination of the F-sequences $\{u_n\}$ and $\{v_n\}$.

Consider the sequence $1, \gamma, \gamma^2, \gamma^3, \cdots$. This will be an F-sequence if $\gamma^{n+1} = \gamma^n + \gamma^{n-1}$ for all integers n; that is, for γ such that $\gamma^2 = \gamma + 1$. This equation has solutions which we will denote by $\beta = \frac{1 + \sqrt{5}}{2}$ and $\alpha = \frac{1 - \sqrt{5}}{2}$. Thus, the α-sequence $1, \alpha, \alpha^2, \cdots$ and the β-sequence $1, \beta, \beta^2,$ are F-sequences. These can be extended to include negative integer exponents as well.

Since

$$\begin{vmatrix} \beta & \beta^2 \\ \alpha & \alpha^2 \end{vmatrix} = \beta - \alpha = \sqrt{5} ,$$

every F-sequence can be written as a unique linear combination of the α-sequence and the β-sequence. (Note that $\alpha + \beta = 1$ and $\alpha\beta = -1$.)

In particular this applies to the Fibonacci sequence. From the equations

$$F_1 = 1 = c\alpha + d\beta$$

$$F_2 = 1 = c\alpha^2 + d\beta^2$$

one finds that $c = -1/\sqrt{5}$ and $d = 1/\sqrt{5}$. Thus,

$$F_n = \frac{\beta^n - \alpha^n}{\beta - \alpha} = \frac{\beta^n - \alpha^n}{\sqrt{5}}$$

The F-sequence with $L_1 = 1$ and $L_2 = 3$ is known as the Lucas sequence. In the case of the Lucas sequence,

$$L_n = \beta^n + \alpha^n \; .$$

The α- and β-sequences can be used to prove many well-known relations involving Fibonacci numbers, Lucas numbers, and general F-sequences:

1. Since

$$F_n = \frac{\beta^n - \alpha^n}{\beta - \alpha}$$

and $L_n = \beta^n + \alpha^n$ then it follows immediately that

since $$F_n \cdot L_n = \frac{\beta^{2n} - \alpha^{2n}}{\beta - \alpha} = F_{2n} \; .$$

2. Since $\beta^{n+1} + \beta^{n-1} = \beta^n(\beta + \beta^{-1}) = \beta^n(\beta - \alpha)$ and $\alpha^{n+1} + \alpha^{n-1} = \alpha^n(\alpha - \beta)$, it follows that $\beta^{n+1} - \alpha^{n+1} + \beta^{n-1} - \alpha^{n-1} = (\beta - \alpha)(\beta^n + \alpha^n)$; thus $L_n = F_{n+1} + F_{n-1}$. Also, $L_{n+1} + L_{n-1} = 5F_n$ can be similarly shown.

3. Let $\{u_n\}$ be an F-sequence, such that $u_n = c\alpha^n + d\beta^n$. Then the determinant

$$\begin{vmatrix} u_{n+1} & u_n \\ u_n & u_{n-1} \end{vmatrix}$$

can be simplified as follows:

$$\begin{vmatrix} u_{n+1} & u_n \\ u_n & u_{n-1} \end{vmatrix} = \begin{vmatrix} c\alpha^{n+1} + d\beta^{n+1} & c\alpha^n + d\beta^n \\ c\alpha^n + d\beta^n & c\alpha^{n-1} + d\beta^{n-1} \end{vmatrix}$$

$$= cd \begin{vmatrix} \alpha^{n+1} & \beta^n \\ \alpha^n & \beta^{n-1} \end{vmatrix} + cd \begin{vmatrix} \beta^{n+1} & \alpha^n \\ \beta^n & \alpha^{n-1} \end{vmatrix}$$

$$= (-1)^{n+1} 5cd \; .$$

In particular,

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n .$$

4. $F^2_{n+p} - F^2_{n-p} = F_{2n} \cdot F_{2p}$ for all p and n. Consider $F_{n+p} + F_{n-p}$. Then,

$$F_{n+p} + F_{n-p} = \frac{\beta^{n+p} - \alpha^{n+p}}{\beta - \alpha} + \frac{\beta^{n-p} - \alpha^{n-p}}{\beta - \alpha}$$

$$= \frac{\beta^n(\beta^p + \beta^{-p}) - \alpha^n(\alpha^p + \alpha^{-p})}{\beta - \alpha}$$

$$= \frac{(\beta^p + \beta^{-p})[\beta^n + (-1)^{p+1}\alpha^n]}{\beta - \alpha} \quad \text{since } \alpha^{-p} = (-1)^p\beta^p$$

$$= \frac{[\beta^n + (-1)^{p+1}\alpha^n][\beta^p + (-1)^p\alpha^p]}{\beta - \alpha}$$

Therefore, if p is even, $F_{n+p} + F_{n-p} = F_n \cdot L_p$ and if p is odd, $F_{n+p} + F_{n-p} = L_n \cdot F_p$.

Also, $F_{n+p} - F_{n-p} = L_n \cdot F_p$ for p even and $F_{n+p} - F_{n-p} = F_n \cdot L_p$ for p odd. Thus, $F^2_{n+p} - F^2_{n-p} = F_{2n} \cdot F_{2p}$ for all p and n.

5. Let us simplify $F_3 + F_6 + \cdots + F_{3n}$. Since the α-sequence and the β-sequence are also geometric sequences it follows that

$$\beta^3 + \cdots + \beta^{3n} = \frac{\beta^3(\beta^{3n} - 1)}{\beta^3 - 1}$$

and

$$\alpha^3 + \cdots + \alpha^{3n} = \frac{\alpha^3(\alpha^{3n} - 1)}{\alpha^3 - 1} .$$

Thus,

$$F_3 + F_6 + \cdots + F_{3n} = \frac{-\beta^{3n} + \alpha^{3n} + \beta^3 - \alpha^3 - \beta^{3n+3} + \alpha^{3n+3}}{(-\alpha^3 - \beta^3)(\beta - \alpha)}$$

$$= \frac{F_{3n+3} + F_{3n} - F_3}{L_3} .$$

6. As another example consider $F_1 + 2F_2 + \cdots + nF_n$, n positive. Now

$$\beta + 2\beta^2 + \cdots + n\beta^n = \beta\left[\frac{n\beta^{n-1} - \beta^n + 1}{\alpha^2}\right]$$

$$= n\beta^{n+2} - \beta^{n+3} + \beta^3$$

since

$$1 + 2x + \cdots + nx^{n-1} = \frac{d}{dx}\left[\frac{x(x^n - 1)}{(x - 1)}\right]$$

Also, $\alpha + 2\alpha^2 + \cdots + n\alpha^n = n\alpha^{n+2} - \alpha^{n+3} + \alpha^3$. Therefore, $F_1 + 2F_2 + \cdots + nF_n = nF_{n+2} - F_{n+3} + F_3$. Note that a similar result holds for a general F-sequence.

7. Let us consider some results that utilize the binomial theorem. Since

$$\beta^n = (1 - \alpha)^n = \sum_{j=0}^{n} \binom{n}{j} (-1)^j \alpha^j$$

and

$$\alpha^n = (1 - \beta)^n = \sum_{j=0}^{n} \binom{n}{j} (-1)^j \beta^j \quad ,$$

it follows that

$$\beta^n - \alpha^n = \sum_{j=0}^{n} \binom{n}{j} (-1)^{j+1} (\beta^j - \alpha^j) \quad ;$$

hence,

$$F_n = \sum_{j=0}^{n} \binom{n}{j} (-1)^{j+1} F_j \quad .$$

Also,

$$L_n = \sum_{j=0}^{n} \binom{n}{j} (-1)^j L_j$$

8. Again using the binomial theorem,

$$\alpha^{2n} = (1+\alpha)^n = \sum_{j=0}^{n} \binom{n}{j} \alpha^j$$

and

$$\beta^{2n} = (1+\beta)^n = \sum_{j=0}^{n} \binom{n}{j} \beta^j \quad .$$

Therefore

$$F_{2n} = \sum_{j=0}^{n} \binom{n}{j} F_j \quad ;$$

also

$$L_{2n} = \sum_{j=0}^{n} \binom{n}{j} L_j \quad .$$

If $\{u_n\}$ is a general F-sequence, it also follows that

$$u_{2n} = \sum_{j=0}^{n} \binom{n}{j} u_j \quad .$$

9. As a final example to illustrate the usefulness of the α- and β-sequences in establishing Fibonacci relations we will derive the result

$$F_n = F_{n-p+1} F_p + F_{n-p} F_{p-1}$$

for all n and p. First, from

$$\beta^p \beta^{n-p-1} + \beta^{p-1}\beta^{n-p} = \beta^{n-1} + \beta^{n-1} = \beta^n(\beta - \alpha)$$

and

$$\beta^p \alpha^{n-p+1} + \beta^{p-1}\alpha^{n-p} = 0$$

we obtain

$$\beta^n = \beta^p F_{n-p+1} + \beta^{p-1} F_{n-p} .$$

Similarly, one can show that

$$\alpha^n = \alpha^p F_{n-p+1} + \alpha^{p-1} F_{n-p} .$$

It then follows that $F_n = F_p F_{n-p+1} + F_{p-1} F_{n-p}$ and if $\{u_n\}$ is an F-sequence, then

$$u_n = u_p F_{n-p+1} + u_{p-1} F_{n-p} .$$

Note that if $q = n - p + 1$, then $u_{p+q-1} = u_p F_q + u_{p-1} F_{q-1}$.

Since $\beta^n - \alpha^n = \sqrt{5}\, F_n$ and $\beta^n + \alpha^n = L_n$, it follows that

$$\beta^n = \frac{L_n + \sqrt{5}\, F_n}{2}$$

and

$$\alpha^n = \frac{L_n - \sqrt{5}\, F_n}{2} .$$

HINTS TO BEGINNERS' CORNER PROBLEMS

(See page 59)

1.1 Examine $\frac{n}{p}$.

1.2 Use identity III.

1.3 Notice that p, $p + 1$, $p + 2$ are three consecutive integers. Since $p > 3$ is an odd prime, $p + 1$ is even. Why must $p + 1$ be a multiple of 3?

1.4 $2^{5\cdot 7} - 1 = (2^5)^7 - (1)^7 = (2^5 - 1)\,[(2^5)^6 + (2^5)^5 + \cdots + (2^5) + 1]$.

1.5 If N is composite, then by T1 it must have a prime factor p. This factor must be one of the following: $2, 3, 5, 7, \cdots, p_n$. Thus $p|N$ and $p|(2\cdot 3\cdot 5\cdots p_n)$.