ADVANCED PROBLEMS AND SOLUTIONS

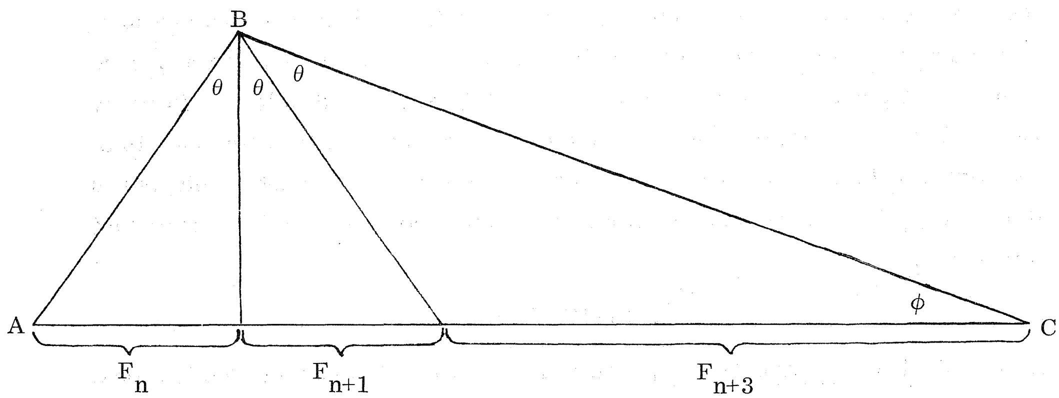
EDITED BY VERNER E. HOGGATT, JR., SAN JOSE STATE COLLEGE

H-19 *Proposed by Charles R. Wall, Texas Christian University, Ft. Worth, Texas.*

In the triangle below [drawn for the case (1, 1, 3)], the trisectors of angle, B, divide side, AC, into segments of length F_n, F_{n+1}, F_{n+3}. Find:

(i) $\lim_{n \to \infty} \theta$

(ii) $\lim_{n \to \infty} \varphi$



H-20 *Proposed by Verner E. Hoggatt, Jr., and Charles H. King, San Jose State College, San Jose, California.*

If $Q = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, show $D(e^{Q^n}) = e^{L_n}$,

where D(A) is the determinant of matrix A and L_n is the n^{th} Lucas number.

H-21 *Proposed by Francis D. Parker, University of Alaska, College, Alaska*

Find the probability, as n approaches infinity, that the n^{th} Fibonacci number, F(n), is divisible by another Fibonacci number ($\neq F_1$ or F_2).

H-22 *Proposed by Verner E. Hoggatt, Jr.*

$$\text{If } P(x) = \prod_{i=1}^{\infty} (1 + x^{F_i}) = \sum_{n=0}^{\infty} R(n)\, x^n \quad ,$$

then show

(i) $$R(F_{2n} - 1) = n$$

(ii) $$R(N) > n \text{ if } N > F_{2n} - 1 \quad .$$

(See first paper of this issue.)

H-23 *Proposed by Malcolm H. Tallman, Brooklyn, New York*

1, 3, 21, and 55 are Fibonacci numbers. Also, they are triangular numbers. What is the next higher number that is common to both series?

SOLUTIONS

H-3 Show $F_{2n-2} < F_n^2 < F_{2n-1}$, $\quad n \geq 3 \quad ;$

$$F_{2n-1} < L_{n-1}^2 < F_{2n}, \quad n \geq 4 \quad ,$$

where F_n and L_n are the n^{th} Fibonacci and Lucas numbers, respectively.

Solution by Francis D. Parker, University of Alaska.

The identities $F^2(n) = F(2n - 2) + F^2(n - 2)$

and $F^2(n) = F(2n - 1) - F^2(n - 1)$

are valid for $n \geq 3$ and can be proved from the explicit formulas for $F(n)$. From these it follows that $F(2n - 2) < F^2(n) < F(2n - 1)$, $n \geq 3$. Again, from the explicit formulas for $L(n)$ and $F(n)$ it is possible to prove the identities $L^2(n - 1) = F(2n - 1) + F(2n - 3) + 2(-1)^{n+1}$ and $L^2(n - 1) = F(2n) - F(2n - 4) + 2(-1)^{n+1}$. From these it follows that $F(2n - 1) < L^2(n - 1) < F(2n)$, $(n \geq 4)$. This problem was also solved by Dov Jarden, Jerusalem, Israel.

H-4 Prove the identity:

$$F_{r+1} F_{s+1} F_{t+1} + F_r F_s F_t - F_{r-1} F_{s-1} F_{t-1} = F_{r+s+t}$$

Are there any restrictions on the integral subscripts?

Solution by J. L. Brown, Jr., Pennsylvania State University, State College, Pennsylvania

We shall prove the assertion under the subscript restrictions, $r \geq -1$, $s \geq -1$, $t \geq -1$, where $F_{-2} = -1$, $F_{-1} = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \leq 0$. The proof is by an induction on n, where $n = r + s + t$. To show that the result holds for $n = 1$ and $n = 2$, a symmetry argument shows that it suffices to verify the result for the nine triples $(r, s, t) = (1, 0, 0)$, $(-1, 1, 1)$, $(3, -1, -1)$, $(-1, 2, 0)$, $(2, 0, 0)$, $(1, 1, 0)$, $(-1, 3, 0)$, $(2, -1, 1)$ and $(4, -1, 1)$.

Now assume as an induction hypothesis that the result has been proved for all n satisfying $n \leq k$, where $k \geq 2$. Then consider any triple (r, s, t) such that $r + s + t = k + 1$. Assume without loss of generality that $r = \max(r, s, t)$. Then $r \geq 1$ and

$$
\begin{aligned}
\Delta_{k+1} \equiv\ & F_{r+1} F_{s+1} F_{t+1} + F_r F_s F_t - F_{r-1} F_{s-1} F_{t-1} \\
=\ & (F_r F_{s+1} F_{t+1} + F_{r-1} F_{s+1} F_{t+1}) \\
& + (F_{r-1} F_s F_t + F_{r-2} F_s F_t) \\
& - (F_{r-2} F_{s-1} F_{t-1} + F_{r-3} F_{s-1} F_{t-1})
\end{aligned}
$$

But

$$F_r F_{s+1} + F_{t+1} + F_{r-1} F_s F_t - F_{r-2} F_{s-1} F_{t-1} = F_{r-1+s+t}$$

by the induction hypothesis applied to the triple $(r-1, s, t)$, which has the sum $r-1+s+t = k$. Similarly

$$F_{r-1} F_{s+1} F_{t+1} + F_{r-2} F_s F_t - F_{r-3} F_{s-1} F_{t-1} = F_{r-2+s+t}$$

by the induction hypothesis applied to the triple $(r-2, s, t)$, which has the sum $r-2+s+t = k-1$. Thus

$$\Delta_{k+1} = F_{r-1+s+t} + F_{r-2+s+t} = F_{r+s+t}$$

as required and the result follows by induction.

H-5 (i) If $$\left[{}_m F_n\right] = \frac{(F_m F_{m-1} \cdots F_1)}{(F_n F_{n-1} \cdots F_1)(F_{m-n} F_{m-n-1} \cdots F_1)}$$

then

$$2\left[{}_m F_n\right] = \left[{}_{m-1} F_n\right] L_n + \left[{}_{m-1} F_{n-1}\right] L_{m-n} ,$$

where F_n and L_n are the n^{th} Fibonacci and n^{th} Lucas numbers, respectively.

(ii) Show that this generalized binomial coefficient $\left[{}_m F_n\right]$ is always an integer.

Solution by J. L. Brown, Jr.

(i) The identity $L_n F_{m-n} + L_{m-n} F_n = 2F_m$ for $m \geq n \geq 0$ is easily verified by induction. Multiplying both sides of the identity by $\left[{}_m F_n\right] \big/ F_m$ then gives the required relation $L_n\left[{}_{m-1} F_n\right] + L_{m-n}\left[{}_{m-1} F_{n-1}\right] = 2\left[{}_m F_n\right]$.

From the expression for F_m, it follows that

$$F_m = \alpha^{m-n} F_n + \beta^n F_{m-n} \quad \text{for } m \geq n .$$

Then

$$(*) \qquad \left[{}_m F_n\right] = \frac{\left[{}_m F_n\right]}{F_m} \cdot F_m = \alpha^{m-n}\left[{}_{m-1} F_{n-1}\right] + \beta^n \left[{}_{m-1} F_n\right] .$$

but $\left[{}_m F_n\right] = \left[{}_m F_{n-m}\right]$. If we replace n by $m-n$ on the right-hand side of (*), then we have

$$(**) \qquad \left[{}_m F_n\right] = \alpha^n \left[{}_{m-1} F_{m-n-1}\right] + \beta^{m-n}\left[{}_{m-1} F_{m-n}\right] .$$

However, $\left[{}_{m-1}F_{m-n-1}\right] = \left[{}_{m-1}F_n\right]$ and $\left[{}_{m-1}F_{m-n}\right] = \left[{}_{m-1}F_{n-1}\right]$, so that adding (*) and (**) yields

$$2\left[{}_m F_n\right] = (\alpha^{m-n} + \beta^{m-n})\left[{}_{m-1} F_{n-1}\right] + (\alpha^n + \beta^n)\left[{}_{m-1} F_n\right]$$

$$= L_{m-n}\left[{}_{m-1}F_{n-1}\right] + L_n\left[{}_{m-1}F_n\right] \quad \text{as required.}$$

(ii) A proof of the second part which makes use of relation (*) can be found on p. 45 of D. Jarden's "Recurring Sequences."

H-6 Determine the last three digits, in base seven, of the millionth Fibonacci number. (Series: $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, etc.)

Solution by Brother U. Alfred, St. Mary's College, Calif.

The last three digits base seven may be determined if we find the residue modulo seven cubed of the millionth Fibonacci number.

Seven has a period of 16 and 7^3 has a period of $7^2 \times 16 = 784$.

In 1,000,000, there are a number of complete periods and a partial period of 400.

For a period of the form $2^m(2\lambda + 1)$ where $m \geq 2$, there is a zero at the half-period of 392. Also, for a prime or a power of a prime, the adjacent terms are congruent to -1 modulo the power of the prime. Hence we know that we have the following series of values:

n	Residue (modulo 343)
392	0
393	342
394	342
395	341
397	340
397	338
398	335
399	330
400	322

This expressed to base seven is $(440)_7$, so that these are the last three digits of the millionth Fibonacci number expressed in base 7.

H-7 If F_n is the n^{th} Fibonacci number find $\lim_{n \to \infty} \sqrt[n]{F_n} = L$ and show that

$$\sqrt[2n]{\sqrt{5}\, F_{2n}} < L < \sqrt[2n+1]{\sqrt{5}\, F_{2n+1}} \quad \text{for } n \geq 2.$$

Solution by John L. Brown, Jr.

Let $a = (1 + \sqrt{5})/2$. Then, it is well-known (see, e.g., pp. 22–23 of "Fibonacci Numbers" by N. N. Vorobév) that

$$\left| F_n - \frac{a^n}{\sqrt{5}} \right| < \frac{1}{2} \text{ for all } n \geq 1 .$$

Therefore $F_n = \frac{a^n}{\sqrt{5}} + \theta_n$, where $|\theta_n| < \frac{1}{2}$ and $\sqrt[n]{F_n} = \sqrt[n]{\frac{a^n}{\sqrt{5}} + \theta_n}$.
But for $n \geq 1$,

$$\frac{a}{\sqrt[n]{2}\ \sqrt[2n]{5}} = \sqrt[n]{\frac{a^n}{2\sqrt{5}}} < \sqrt[n]{\frac{2a^n - \sqrt{5}}{2\sqrt{5}}} = \sqrt[n]{\frac{a^n}{\sqrt{5}} - \frac{1}{2}} < \sqrt[n]{\frac{a^n}{\sqrt{5}} + \theta_n}$$

$$< \sqrt[n]{\frac{a^n}{\sqrt{5}} + \frac{1}{2}} = \sqrt[n]{\frac{2a^n + 1}{\sqrt{5}}} < \sqrt[n]{\frac{3a^n}{2\sqrt{5}}} = \frac{a}{\sqrt[n]{\frac{2\sqrt{5}}{3}}} .$$

Taking $\lim_{n \to \infty}$, we find

$$L = \lim_{n \to \infty} \sqrt[n]{\frac{a^n}{\sqrt{5}} + \theta_n} = a .$$

Thus $L = a = \frac{1 + \sqrt{5}}{2}$.

Now, let $b = \frac{1 - \sqrt{5}}{2}$ so that $b < 0$. Then, since $\sqrt{5}\, F_n = a^n - b^n$ for $n \geq 1$, we have

$$\sqrt[2n]{\sqrt{5} F_{2n}} = \sqrt[2n]{a^{2n} - b^{2n}} < a < \sqrt[2n+1]{a^{2n+1} - b^{2n+1}} = \sqrt[2n+1]{\sqrt{5}\, F_{2n+1}}$$

thus the desired inequality follows for all $n \geq 1$ on noting that $L = a$.

Also solved by Donna Seaman.

H-8 Prove

$$\begin{vmatrix} F_n^2 & F_{n+1}^2 & F_{n+1}^2 \\ F_{n+1}^2 & F_{n+2}^2 & F_{n+3}^2 \\ F_{n+2}^2 & F_{n+3}^2 & F_{n+4}^2 \end{vmatrix} = 2(-1)^{n+1} ,$$

where F_n is the n^{th} Fibonacci number.

Solution by John Allen Fuchs and Joseph Erbacher, University of Santa Clara, Santa Clara, California

The squares of the Fibonacci numbers satisfy the linear homogeneous recursion relationship $F_{n+3}^2 = 2F_{n+2}^2 + 2F_{n+1}^2 - F_n^2$. (See H. W. Gould, Generating Functions for Products of Powers of Fibonacci Numbers, this Quarterly, Vol. 1, No. 2, p. 2.)

We may use this recursion formula to substitute for the last row of the given determinant, D_n, and then apply standard row operations to get

$$D_n = \begin{vmatrix} F_n^2 & F_{n+1}^2 & F_{n+2}^2 \\ F_{n+1}^2 & F_{n+2}^2 & F_{n+3}^2 \\ 2F_{n+1}^2 + 2F_n^2 - F_{n-1}^2 & 2F_{n+2}^2 + 2F_{n+1}^2 - F_n^2 & 2F_{n+3}^2 + 2F_{n+2}^2 - F_{n+1}^2 \end{vmatrix}$$

$$= \begin{vmatrix} F_n^2 & F_{n+1}^2 & F_{n+2}^2 \\ F_{n+1}^2 & F_{n+2}^2 & F_{n+3}^2 \\ -F_{n-1}^2 & -F_n^2 & -F_{n+1}^2 \end{vmatrix} = -D_{n-1} .$$

It follows immediately by induction that $D_n = (-1)^{n-1} D_1$. Since $D_1 = 2$, $D_n = 2(-1)^{n-1} = 2(-1)^{n+1}$.

Also solved by Marjorie Bicknell and Dov Jarden.

Continued from p. 80, "Elementary Problems and Solutions"

Then

$$F_{k+2}p^{k+1} + F_{k+1}p^{k+2} = (F_{k+1} + F_k)p^{k+1} + (F_k + F_{k-1})p^{k+2}$$

$$= p(F_{k+1}p^k + F_k p^{k+1}) + p^2(F_k p^{k+1} + F_{k-1}p^k) .$$

But

$$p(F_{k+1}p^k + F_k p^{k+1}) + p^2(F_k p^{k-1} + F_{k-1}p^k) \equiv p + p^2 (\text{mod } p^2 + p - 1).$$

Since $F_{k+1}p^k + F_k p^{k+1}$ and $F_k p^{k-1} + F_{k-1}p^k$ are both congruent to 1 (mod $p^2 + p - 1$) by the induction hypothesis and $p + p^2 \equiv 1$ (mod $p^2 + p - 1$), the desired result follows by induction on n.

Also solved by Marjorie R. Bicknell and Donna J. Seaman.