A PRIMER FOR THE FIBONACCI SEQUENCE — PART III

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1. INTRODUCTION

The algebra of vectors and matrices will be further pursued to derive some more Fibonacci identities.

2. THE ALGEBRA OF (TWO-DIMENSIONAL) VECTORS

The two-dimensional vector, V, is an ordered pair of elements, called scalars, of a field: (The real numbers, for example, form a field.)

$$V = (a, b) .$$

The zero vector, ϕ, is a vector whose elements are each zero (i.e., $a = 0$ and $b = 0$).

Two vectors, $U = (a,b)$ and $V = (c,d)$, are equal if and only if $a = c$ and $b = d$, that is (iff) their corresponding elements are equal.

The vector W, which is the product of a scalar, k, and a vector, $U = (a,b)$, is

$$W = kU = (ka, kb) = Uk .$$

We see that if $k = 1$, then $kU = U$. We shall define the additive inverse of U, -U, by $-U = (-1)U$.

The vector W, which is the vector sum of two vectors $U = (a,b)$ and $V = (c,d)$ is

$$W = U + V = (a,b) + (c,d) = (a + c, b + d) .$$

The vector $W = U - V$ is

$$W = U - V = U + (-V)$$

which defines subtraction.

The only binary multiplicative operation between two vectors, $U = (a,b)$ and $V = (c,d)$, considered here is the scalar or inner product, $U \cdot V$,

$$U \cdot V = (a,b) \cdot (c,d) = ac + bd ,$$

which is a scalar.

3. A GEOMETRIC INTERPRETATION OF A TWO-DIMENSIONAL VECTOR

One interpretation of the vector, $U = (a,b)$, is a directed line segment from the origin $(0,0)$ to the point (a,b) in a rectangular coordinate system. Every vector, except the zero vector, ϕ, will have the direction from the origin $(0,0)$ to the point (a,b) and a magnitude or length, $U = \sqrt{a^2 + b^2}$. The zero vector, ϕ, has a zero magnitude and no defined direction.

The inner or scalar product of two vectors, $U = (a,b)$ and $V = (c,d)$ can be shown to equal

$$U \cdot V = |U||V| \cos \theta \,,$$

where θ is the angle between the two vectors.

4. TWO-BY-TWO MATRICES AND TWO-DIMENSIONAL VECTORS

If $U = (a,b)$ is written as $(a\ b)$, then U is a 1x2 matrix which we shall call a row-vector. If $U = (a,b)$ is written $\begin{pmatrix} a \\ b \end{pmatrix}$, then U is a 2x1 matrix, which we shall call a column-vector.

The matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

for example, can be considered as two row vectors $R_1 = (a\ b)$ and $R_2 = (c\ d)$ in special position or, as two column vectors, $C_1 = \begin{pmatrix} a \\ c \end{pmatrix}$ and $C_2 = \begin{pmatrix} b \\ d \end{pmatrix}$ in special position.

The product W of a matrix A and a column-vector $X = \begin{pmatrix} x \\ y \end{pmatrix}$ is a column-vector, X',

$$X' = AX = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} .$$

Thus matrix A, operating upon the vector, X, yields another vector, X'. The zero vector, $\phi = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, is transformed into the zero vector again. In general, the direction and magnitude of vector, X, are different from those of vector X'. (See "Beginners' Corner" this issue.)

5. THE INVERSE OF A TWO-BY-TWO MATRIX

If the determinant, d(A), of a two-by-two matrix, A, is non-zero, then there exists a matrix, A^{-1}, the inverse of matrix A, such that

$$A^{-1}A = AA^{-1} = I \quad .$$

From the equation $AX = X'$ or pair of equations

$$ax + by = x' \qquad cx + dy = y'$$

one can solve for the variables x and y in terms of a, b, c, d; and x', y' provided $D(A) = ad - bc \neq 0$. Suppose this has been done so that (let $D = D(A) \neq 0$)

$$\frac{d}{D}x' - \frac{b}{D}y' = x$$

$$\frac{-c}{D}x' + \frac{a}{D}y' = y \quad .$$

Thus the matrix B, such that $BX' = X$ is given by

$$B = \begin{pmatrix} \frac{d}{D} & \frac{-b}{D} \\ \frac{-c}{D} & \frac{a}{D} \end{pmatrix}, \quad D \neq 0 \quad .$$

It is easy to verify that $BA = AB = I$. Thus B is A^{-1}, the inverse matrix to matrix A. The inverse of the Q matrix is $Q^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$.

6. FIBONACCI IDENTITY USING THE Q MATRIX

Suppose we prove, recalling $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}, \text{ that } F_1 + F_2 + \cdots + F_n = F_{n+2} - 1 \quad .$$

It is easy to establish by induction that

$$(I + Q + Q^2 + \cdots + Q^n)(Q - I) = Q^{n+1} - I \quad .$$

If $Q - I$ has an inverse $(Q - I)^{-1}$, then multiplying on each side yields

$$I + Q + Q^2 + \cdots + Q^n = (Q^{n+1} - I)(Q - I)^{-1} .$$

It is easy to verify that $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ satisfies the matrix equation $Q^2 = Q + I$. Thus $(Q - I)Q = Q^2 - Q = I$ and $(Q - I)^{-1} = Q$. Therefore

$$Q + \cdots + Q^n = Q^{n+2} - (Q + I) = Q^{n+2} - Q^2 .$$

Equating elements in the upper right (in the above matrix equation) yields

$$F_1 + F_2 + \cdots + F_n = F_{n+2} - F_2 = F_{n+2} - 1 .$$

7. THE CHARACTERISTIC POLYNOMIAL OF MATRIX A

In Section 4, we discussed the transformation $AX = X'$. Generally the direction and magnitude of vector, X, are different from those of vector, X'. If we ask which vectors X have their directions unchanged, we are led to the equation

$$AX = \lambda X, \quad (\lambda \text{, a scalar}) .$$

This can be rewritten $(A - \lambda I)X = 0$. Since we want $|X| \neq 0$, the only possible solution occurs when $D(A - \lambda I) = 0$. This last equation is called the characteristic equation of matrix A. The values of λ are called characteristic values of eigenvalues and the associated vectors are the characteristic vectors of matrix A. The characteristic polynomial of A is $D(A - \lambda I)$.

The characteristic polynomial for the Q matrix is $\lambda^2 - \lambda - 1 = 0$. The Hamilton-Cayley theorem states a matrix satisfies its own characteristic equation, so that for the Q matrix

$$Q^2 - Q - I = 0 .$$

8. SOME MORE IDENTITIES

Let $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, which satisfies $Q^2 = Q + I$, thus (remembering $Q^0 = I$)

$$(1) \qquad Q^{2n} = (Q^2)^n = (Q + I)^n = \sum_{i=0}^{n} \binom{n}{i} Q^i$$

Equating elements in the upper right yields

$$F_{2n} = \sum_{i=0}^{n} \binom{n}{i} F_i .$$

(Compare with problems H-18 and B-4).

From (1)

$$Q^p Q^{2n} = \sum_{i=0}^{n} \binom{n}{i} Q^{i+p} ,$$

which gives

$$F_{2n+p} = \sum_{i=0}^{n} \binom{n}{i} F_{i+p} \quad (n \geq 0, \text{ integral } p)$$

From part II, $Q^n = F_n Q + F_{n-1} I$,

$$Q^{mn+p} = \sum_{i=0}^{m} \binom{m}{i} Q^{i+p} F_n^i F_{n-1}^{m-i} .$$

Equating elements in upper right of the above matrix equation gives

$$F_{mn+p} = \sum_{i=0}^{m} \binom{m}{i} F_{i+p} F_n^i F_{n-1}^{m-i} ,$$

with $m \geq 0$, any integral p and n.
(See the result p. 38, line 12, issue 2, and H-13).

HAVE YOU SEEN??

Melvin Hochster, "Fibonacci-Type Series and Pascal's Triangle," Particle, Vol. IV, No. 1, Summer 1962, pp. 14–28. (Written while author was a sophomore at Harvard University, but the work was done while he was a senior student at Stuyvesant High School, New York, New York.)

Particle is a quarterly by and for science students with editorial and publishing offices located at 2531 Ridge Road, Berkeley 9, California. The present editor is Steve Kahn.

A. Hamilton Bolton, The Elliott Wave Principle—A Critical Appraisal, Bolton—Tremblay and Company, Montreal 2, Canada, Chap. IX, pp. 61–67.

This is an interesting application of the Fibonacci Sequence to business cycles, and will merit some interest.