ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY VERNER E. HOGGATT, JR., SAN JOSE STATE COLLEGE

Send all communications concerning Advanced Problems and Solutions to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, California. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-24 *Proposed by the late Morgan Ward, California Institute of Technology, Pasadena, California.*

Let $\phi_n(x) = x + x^2/2 + \cdots + x^n/n$, and let $k(x) \equiv k_p(x) = (x^{p-1} - 1)/p$, where p is an odd prime greater than 5. (The function k(x) is called the "quotient of Fermat" in the literature.) Let $P = P_p$ be the rank of apparition of p in the sequence $0, 1, 1, 2, 3, 5, \cdots, F_n$, (so $P_{13} = 7$, $P_7 = 8$ and so on).

Then

$$F_P \equiv 0 \bmod p^2$$

if and only if

$$\phi_{(p-1)/2}(5/9) \equiv 2k(3/2) \bmod p \quad .$$

H-25 *Proposed by Joseph Erbacker and John A. Fuchs, University of Santa Clara, and F.D. Parker, Suny, Buffalo, N.Y.*

Prove:

$$D_n = \left| a_{ij} \right| = 36, \quad \text{for all } n \quad ,$$

where

$$a_{ij} = F^3_{n+i+j-2} \quad (i,j = 1,2,3) \quad .$$

H-26 *Proposed by Leonard Carlitz, Duke University, Durham, N.C.*

Let $R_k = (b_{rs})$, where $b_{rs} = \binom{r-1}{k+1-s}$, then show $R_k^n = (a_{rs})$ such that

$$a_{rs} = \sum_{j=0}^{s-1} \binom{r-1}{j}\binom{k+1-r}{s-1-j} F_{n-1}^{k+2-r-s+j} F_n^{r+s-2-2j} F_{n+1}^{j} \quad .$$

H-27 *Proposed by Harlan L. Umansky, Emerson High School, Union City, N.J.*

Show that

$$F_k^3 = \sum_{j=1}^{k-2} (-1)^{j+1} F_j F_{3k-3j} + (-1)^k F_{k-3} \,, \quad k \geq 4 \ .$$

H-28 *Proposed by H. W. Gould, West Virginia University, Morgantown, W. Va.*

Let $C_j(r, n)$ be the number of numbers, to the base r ($r \geq 2$) with at most n digits, and the sum of the digits equal to j.

Sum the series:

$$\sum_{j=0}^{\infty} C_j(r, n)\, a^j b^{rn-n-j} \quad .$$

SOLUTIONS

TRINOMIAL COEFFICIENTS

H-9 *Proposed by Olga Taussky, California Institute of Technology, Pasadena, Calif.*

Find the numbers $a_{n,r}$, where $n \geq 0$ and r are integers, for which the relations

$$a_{n,r} + a_{n,r-1} + a_{n,r-2} = a_{n+1,r}$$

and

$$a_{o,r} = \delta_{o,r} = \begin{cases} 0 & r \neq 0 \\ 1 & r = 0 \end{cases}$$

hold.

Solution by the proposer.

It can be shown that $a_{n,r}$ is the coefficient of x^r in the expansion of $(1 + x + x^2)^n$.

This is certainly true for $n = 0$ and it follows for $n > 0$ by using the generating functions

$$\sum_r a_{n,r} x^r \quad .$$

For, multiplying this sum by $(1 + x + x^2)$ and using the recurrence relations it follows that

$$(1 + x + x^2) \sum_r a_{n,r} x^r = \sum_r a_{n+1,r} x^r .$$

This proves the assertion.

SOME FIBONACCI SUMS

H-10 *Proposed by R.L. Graham, Bell Telephone Laboratories, Murray Hill, New Jersey*

Show that
$$\sum_{n=1}^{\infty} \frac{1}{F_n} = 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{F_n F_{n+1} F_{n+2}}$$

Solution by Leonard Carlitz, Duke University, Durham, N. C.

$$\sum_2^{\infty} \frac{1}{F_n} - \sum_2^{\infty} \frac{F_n}{F_{n-1}F_{n+1}} = \sum_2^{\infty} \frac{F_{n-1}F_{n+1} - F_n^2}{F_{n-1}F_nF_{n+1}} = \sum_2^{\infty} \frac{(-1)^n}{F_{n-1}F_nF_{n+1}} ,$$

$$\sum_2^{\infty} \frac{F_n}{F_{n-1}F_{n+1}} = \sum_2^{\infty} \frac{F_{n+1} - F_{n-1}}{F_{n-1}F_{n+1}} = \sum_2^{\infty} \left(\frac{1}{F_{n-1}} - \frac{1}{F_{n+1}} \right) = \frac{1}{F_1} + \frac{1}{F_2} = 2 ,$$

$$\sum_1^{\infty} \frac{1}{F_n} = 3 + \sum_2^{\infty} \frac{(-1)^n}{F_{n-1}F_nF_{n+1}} .$$

Also solved by Zvi Dresner.

FIBONACCI AND FOURIER

H-11 *Proposed by John L.Brown,Jr.,Ordnance Research Laboratory, The Pennsylvania State University, University Park, Penna.*

Find the function whose formal Fourier series is

$$f(x) = \sum_{n=1}^{\infty} \frac{F_n \sin nx}{n!} ,$$

where F_n is the nth Fibonacci number.

Solution by Lucile Morton, Santa Clara, Calif.

$$e^{z(\cos x + i \sin x)} = \sum_{n=0}^{\infty} \frac{(\cos x + i \sin x)^n z^n}{n!} = \sum_{n=0}^{\infty} \frac{\cos nx}{n!} z^n + i \sum_{n=0}^{\infty} \frac{\sin nx}{n!} z^n .$$

Therefore

$$e^{z \cos x} \sin(z \sin x) = \sum_{n=0}^{\infty} \frac{\sin nx}{n!} z^n$$

Recalling

$$F_n = \frac{1}{\sqrt{5}} (\alpha^n - \beta^n), \quad \text{where } \alpha = (1 + \sqrt{5})/2 \text{ and } \beta = (1 - \sqrt{5})/2,$$

then

$$f(x) = \frac{1}{\sqrt{5}} \left\{ e^{\alpha \cos x} \sin(\alpha \sin x) - e^{\beta \cos x} \sin(\beta \sin x) \right\}$$

and

$$g(x) = e^{\alpha \cos x} \sin(\alpha \sin x) + e^{\beta \cos x} \sin(\beta \sin x) = \sum_{n=0}^{\infty} \frac{\sin nx}{n!} L_n .$$

Also solved by the proposer.

A CURIOUS SEQUENCE

H-12 *Proposed by D.E. Thoro, San Jose State College, San Jose, Calif.*

Find a formula for the nth term in the sequence:

1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, 21, 22, 24, 25, $\cdots$.

Solution by Malcolm Tallman, Brooklyn, N. Y.

$$N_M \begin{cases} 1, 3, 4, 6, 8, 9, 11, 12 \\ 14, 16, 17, 19, 21, 22, 24, 25 \\ 27, 29, 30, 32, 34, 35, 37, 38 \\ 40, 42, 43, 45, 47, 48, 50, 51 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{cases}$$

Let

$$M = 8m + 1, 2, 3^*, 4, 5, 6, 7, 8$$

$$N_M = 13m + 1, 3, 4^*, 6, 8, 9, 11, 12$$

What is the 19th term? $M = 19 = 8 \times 2 + 3^*$, thus $N_{19} = 13 \times 2 + 4^* = 30$.

Also solved by Maxey Brooke and the proposer.

Editorial Comment: If $T_1 = 1$, $T_2 = 3$, $T_3 = 4$, $T_4 = 6$, $T_5 = 8$, $T_6 = 9$, $T_7 = 11$, $T_8 = 12$, then $T_{8m+k} = 13m + T_k$, $k > 0$, $m = 1, 2, 3, \cdots$.

A MATRIX DERIVED IDENTITY

H-13 *Proposed by H.W. Gould, West Virginia University, Morgantown, W. Va. and Verner E. Hoggatt, Jr., San Jose State College, San Jose, Calif.*

Show that

$$F_n = \sum_{j=0}^{r} \binom{r}{j} F_{k-1}^{r-j} F_k^j F_{n+j-rk} .$$

See p. 65 of "A Primer for the Fibonacci Numbers—Part III," Oct., 1963, Fibonacci Quarterly.

Also solved by Leonard Carlitz and Merritt Elmore.

IDENTITY FOR FIBONACCI CUBES

H-14 *Proposed by David Zeitlin, Minneapolis, Minnesota, and F. D. Parker, University of Alaska, College, Alaska.*

Prove the Fibonacci identity

$$F_{n+4}^3 - 3F_{n+3}^3 - 6F_{n+2}^3 + 3F_{n+1}^3 + F_n^3 = 0 \quad .$$

Solution by Maxey Brooke, Sweeny, Texas.

From "Fibonacci Formulas," page 60, April, 1963, Fibonacci Quarterly, one obtains, from paragraph 3,

(1) $$F_{n+1}^3 + F_n^3 - F_{n-1}^3 = F_{3n}$$

and the corrected version of Jekuthiel Ginsburg's identity there is

(2) $$F_{n+2}^3 - 3F_n^3 + F_{n-2}^3 = F_{3n}$$

Multiplying equation (1) through by 3 and equating the new left side of (1) to the left side of (2) and simplifying yields

$$F_{n+2}^3 - 3F_{n+1}^3 - 6F_n^3 + 3F_{n-1}^3 + F_{n-2}^3 = 0 \quad .$$

Also solved by J. A. H. Hunter, Zvi Dresner and the proposers.

SOME CHOICE IDENTITIES

H-16 *Proposed by H. W. Gould, West Virginia University, Morgantown, W. Va.*

Define the ordinary Hermite polynomials by $H_n = (-1)^n e^{x^2} D^n (e^{-x^2})$.

$$\text{(i)} \quad \sum_{n=0}^{\infty} H_n(x/2) \frac{x^n}{n!} = 1 \quad ,$$

Show that

$$\text{(ii)} \quad \sum_{n=0}^{\infty} H_n(x/2) \frac{x^n}{n!} F_n = 0$$

$$\text{(iii)} \quad \sum_{n=0}^{\infty} H_n(x/2) \frac{x^n}{n!} L_n = 2e^{-x^2} ,$$

where F_n and L_n are the nth Fibonacci and nth Lucas numbers, respectively.

Solution by Zvi Dresner, Tel-Aviv, Israel.

$$\text{(i)} \quad \sum_{n=0}^{\infty} H_n\left(\frac{x_0}{2}\right)\frac{x_0^n}{n!} = e^{\frac{x_0^2}{4}} \left\{ \sum_{n=0}^{\infty} \frac{D^n(e^{-x^2})}{n!}\bigg|_{x=x_0/2} \left(-\frac{x_0}{2} - \frac{x_0}{2}\right)^n \right\} .$$

The sum in braces on the right is the expansion of e^{-x^2} about the point $+\frac{x_0}{2}$, with $x = -x_0/2$. Hence

$$\sum_{n=0}^{\infty} H_n\left(\frac{x_0}{2}\right)\frac{x_0^n}{n!} = e^{+x_0^2/4}\left(e^{-x_0^2/4}\right) = 1 .$$

(ii) In the same way $\left(\alpha = \frac{1+\sqrt{5}}{2},\ \beta = \frac{1-\sqrt{5}}{2},\ \text{and } F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)\right)$,

$$\sum_{n=0}^{\infty} H_n\left(\frac{x_0}{2}\right)\frac{x_0^n}{n!} F_n = \frac{1}{\sqrt{5}} e^{x_0^2/4} \left\{ \sum_{n=0}^{\infty} \frac{D^n e^{-x^2}}{n!}\bigg|_{x=x_0/2} \left((-x_0\alpha)^n - (-x_0\beta)^n\right) \right\}$$

$$= \frac{1}{\sqrt{5}} e^{x_0^2/4} \left\{ e^{-\left(\frac{x_0\sqrt{5}}{2}\right)^2} - e^{-\left(\frac{-x_0\sqrt{5}}{2}\right)^2} \right\} = 0.$$

(iii) Similarly $(L_n = \alpha^n + \beta^n)$

$$\sum_{n=0}^{\infty} H_n\left(\frac{x_0}{2}\right)\frac{x_0^n}{n!} L_n = e^{x_0^2/4} \sum_{n=0}^{\infty} \frac{D^n e^{-x^2}}{n!}\bigg|_{x=x_0/2} \left[(-x_0\alpha)^n + (-x_0\beta)^n\right]$$

$$= e^{x_0^2/4}\left(2e^{-5x_0^2/4}\right) = 2e^{-x_0^2} .$$

Also solved by L. Carlitz and the proposer.

Correction to Problem H-20 in the October issue

H-20 (Corrected) *Proposed by Verner E. Hoggatt, Jr. and Charles H. King, San Jose State College, San Jose, California.*

If $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, show $D(e^{Q^n}) = e^{L_n}$,

where $D(A)$ is the determinant of matrix A and L_n is the nth Lucas number.