A PRIMER FOR THE FIBONACCI NUMBERS — PART IV
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1. INTRODUCTION

In the primer, Part III, it was noted that if $V = (x, y)$ is a two-dimensional vector and $A$ is a 2 by 2 matrix, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $V' = AV$ is a two-dimensional vector, $V' = (x', y') = (ax + by, cx + dy)$. Here, $V$ and consequently $V'$, are expressed as column vectors. The matrix $A$ is said to transform, or map, the vector $V$ onto the vector $V'$. The matrix $A$ is called the mapping matrix or transformation matrix.

2. SOME MAPPING MATRICES

The zero matrix, $Z = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, maps every vector $V$ onto the zero vector $\phi = (0,0)$.

The identity matrix, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ maps every vector $V$ onto itself; that is, $IV = V$.

The matrix $B = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ maps vectors $V = (k,-k)$, ($k$ any real number), onto the zero vector $\phi$. Such a mapping as determined by $B$ is called a many-to-one mapping.

If the only vector mapped onto $\phi$ is the vector $\phi$ itself, the mapping is a one-to-one mapping. A matrix $A$ determines a one-to-one mapping of two-dimensional vectors onto two-dimensional vectors if, and only if, $\det A \neq 0$. If $\det A \neq 0$, for each vector $U$, there exists a vector $V$ such that $AV = U$. Note, however, that for matrix $B$ above, $B\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ 2x + 2y \end{pmatrix}$. There is no vector $V$ such that $BV = (0,1)$.

3. GEOMETRIC INTERPRETATIONS OF 2x2 MATRICES AND 2-DIMENSIONAL VECTORS

As in Primer III, the vector $V = (x, y)$ is interpreted as a point in a rectangular coordinate system. Thus the geometric concepts of length, direction, slope and angle are associated with the vector $V$.

A non-zero scalar multiple of the identity matrix, $kI$, maps the vector $U = (a, b)$ onto the vector $V = (ka, kb)$. The length of $V$, $|V|$, is equal to $|k| |U|$. There is no change in slope but if $k < 0$ the sense or direction is reversed.
The matrix \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) maps a vector onto the reflection vector with respect to the line through the origin with slope one. Note that different vectors may be rotated through different angles!

The matrix \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \) preserves the first component of a vector while annihilating the second component. Every vector \( U = (x,y) \) is mapped into a vector on the x-axis.

The matrix \( R = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \) rotates all vectors through the same angle \( \theta \) (theta), in a counterclockwise direction if theta is a positive angle. There is no change in length. This seems to contradict the notion of a matrix having vectors whose slopes are not changed but in this case the characteristic values are complex; thus, there are no real characteristic vectors.

4. THE CHARACTERISTIC VECTORS OF THE Q-MATRIX

The Q matrix \( \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \) does not generally preserve the length of a vector \( U = (x,y) \). Also, different vectors are in general rotated through different angles.

The characteristic equation of the Q matrix is

\[
\lambda^2 - \lambda - 1 = 0
\]

with roots

\[
\lambda_1 = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{1 - \sqrt{5}}{2},
\]

which are the characteristic vectors, or eigenvalues, for Q.

To solve for a pair of corresponding characteristic vectors consider

\[
\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \lambda \begin{pmatrix} x' \\ y' \end{pmatrix}, \quad x'^2 + y'^2 = 0.
\]

Then

\[
(1 - \lambda)x + y = 0.
\]

Thus, a pair of characteristic vectors are

\[
X_1 = (\lambda x, x), \quad |X_1| \neq 0,
\]

with slope

\[
m_1 = \frac{\sqrt{5} - 1}{2} \quad \text{and} \quad X_2 = (\lambda_2 x, x), \quad |X_2| \neq 0,
\]

with slope

\[
m_2 = \frac{\sqrt{5} + 1}{2}.
\]
What happens when the matrix $Q^2$ is applied to the characteristic vectors $X_1$ and $X_2$ of matrix $Q$? Since

$$Q^2 X_1 = Q(QX_1) = Q(\lambda X_1) = \lambda QX_1 = \lambda^2 X_1,$$

clearly $X_1$ is a characteristic vector of the matrix $Q^2$ as well as a characteristic vector of matrix $Q$. The characteristic roots of $Q^2$ are the squares of the characteristic roots of matrix $Q$. In general if $\lambda_1$ and $\lambda_2$ are the characteristic roots of $Q$ then $\lambda_1^n$ and $\lambda_2^n$ are the characteristic roots of $Q^n$. But the characteristic equation for $Q^n$ is

$$\lambda^n - (F_{n+1} + F_{n-1})\lambda^{n-2} + (F_{n+1} F_{n-1} - F_n^2) = 0.$$

Recalling that $L_n = F_{n+1} + F_{n-1}$, $F_{n+1} F_{n-1} - F_n^2 = (-1)^n$, and $L_n^2 = 5F_n^2 + 4(-1)^n$, it follows that, since $\lambda_1 = \alpha = (1 + \sqrt{5})/2$ and $\lambda_2 = \beta = (1 - \sqrt{5})/2,$

$$\alpha^n = \lambda_1^n = (L_n + \sqrt{5}F_n)/2 \quad \text{and} \quad \beta^n = \lambda_2^n = (L_n - \sqrt{5}F_n)/2.$$

5. FIBONACCI AND LUCAS VECTORS AND THE $Q$ MATRIX

Let $U_n = (F_{n+1}, F_n)$ and $V_n = (L_{n+1}, L_n)$ be denoted as Fibonacci and Lucas vectors, respectively. We note

$$|U_n|^2 = F_{n+1}^2 + F_n^2 = F_{2n+1} \quad \text{and} \quad |V_n|^2 = L_{n+1}^2 + L_n^2 = (5F_{n+1}^2 + (-1)^n 4 + 5F_n^2 + (-1)^n 4) = 5(F_{n+1}^2 + F_n^2) = 5F_{2n+1}^2.$$

It is well known that the slopes of the vectors $U_n$ and $V_n$ (the ratios $F_n/F_{n+1}$ and $L_n/L_{n+1}$) approach the slope, $(\sqrt{5} - 1)/2$, of the characteristic vector, $X_1$.

Since $Q^m Q^n = Q^{m+n}$, it is easy to verify that

$$F_{m+1} F_{n+1} + F_m F_n = F_{m+n+1}$$

by equating elements in the upper left in the above matrix equation. In a similar manner it follows that

$$F_{m+1} F_{n+2} + F_m F_{n+1} = F_{m+n+2}$$
$$F_{m+1} F_n + F_m F_{n-1} = F_{m+n}$$

Adding these two equations and using $L_{n+1} = F_{n+2} + F_n$ it follows that

$$F_{m+1} L_{n+1} + F_m L_n = L_{m+n+1}.$$
From the above identities it is easy to verify that

\[
Q^{n+1}V_0 = QV_n = V_{n+1}, \\
Q^{n+1}U_0 = QU_n = U_{n+1}, \\
Q^nV_m = V_{m+n+1}, \\
Q^nU_m = U_{m+n+1}.
\]

6. A SPECIAL MATRIX

Let \( P = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \), then from

\[
L_{n+1} = F_{n+1} + 2F_n, \quad L_n = 2F_{n+1} - F_n, \\
5F_{n+1} = L_{n+1} + 2L_n, \quad 5F_n = 2L_{n+1} - L_n,
\]

it follows that

\[
PU_n = (F_{n+1} + 2F_n, 2F_{n+1} - F_n) = V_n, \\
PV_n = (L_{n+1} + 2L_n, 2L_{n+1} - L_n) = 5U_n
\]

Also

\[
PQ^n = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{pmatrix}
\]

\[
P^2Q^n = 5Q^n,
\]

\[
D\begin{pmatrix} L_{n+1} & L_n \\ L_n & L_{n-1} \end{pmatrix} = D(P)D(Q^n) = 5(-1)^{n+1}
\]

We now discuss two geometric properties of matrix \( P \). Let \( U = (x, y) \),

\[|U|^2 = x^2 + y^2 \neq 0.\]

\[
PU = (x + 2y, 2x - y), \quad |PU|^2 = 5(x^2 + y^2) = 5|U|^2.
\]

Thus matrix \( P \) magnifies each vector length by \( \sqrt{5} \).

If \( \tan \alpha = y/x \), we say \( \alpha = \tan^{-1} y/x \), read "\( \alpha \) is an angle whose tangent is \( y/x \)." Let \( \tan \alpha = y/x \) and \( \tan \beta = (2x - y)/(x + 2y) \). From \( \tan(\alpha + \beta) = (\tan \alpha + \tan \beta)/(1 - \tan \alpha \tan \beta) \) we may now see what effect \( P \) has on the slope of vector \( U = (x, y) \).
Now (recalling \( x^2 + y^2 \neq 0 \) says \( x \) and \( y \) are not both zero at the same
time,) \[
\tan(\alpha + \beta) = \tan \left( \tan^{-1} \frac{y}{x} + \tan^{-1} \frac{2x - y}{x + 2y} \right) = \frac{2(x^2 + y^2)}{x^2 + y^2}.
\]
Thus, since \( x^2 + y^2 \neq 0 \), then
\[
\tan(\alpha + \beta) = 2.
\]

What does this mean? Consider two vectors \( A \) and \( B \), the first inclined at
an angle \( \alpha \) with the positive \( x \)-axis and the second inclined at an angle \( \beta \) with
the positive \( x \)-axis and the angles are measured positively in the counterclock-
wise direction. The angle bisector, \( \psi \), of the angle between vectors \( A \) and
\( B \) is such that \( \alpha - \psi = \psi - \beta \) whether or not \( \alpha \) is greater than \( \beta \) or the other
way around. Solving for \( \psi \) yields
\[
\psi = (\alpha + \beta)/2.
\]
Thus \( \psi \) is the arithmetic average of \( \alpha \) and \( \beta \). Also we note that \( \alpha + \beta = 2\psi \).
The tangent of double the angle is given by
\[
\tan 2\psi = (2 \tan \psi)/(1 - \tan^2 \psi).
\]
Let
\[
\tan \psi = \frac{\sqrt{5} - 1}{2},
\]
then it is an easy exercise in algebra to find \( \tan 2\psi = 2 \), but \( \tan(\alpha + \beta) = 2 \),
therefore we would like to conclude that the angle bisector between vectors \( U \)
and \( PU \) is precisely one whose slope is \( (\sqrt{5} - 1)/2 \), but this is the slope of
\( X_1 \), the characteristic vector of \( Q \). Can you show that \( X_1 \) is also a character-
istic vector of \( P \)?

We have shown

**Theorem 1.** The matrix \( P = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \) maps a vector \( U = (x, y) \) into a
vector \( PU \) such that

(1) \[ |P(U)| = \sqrt{5} |U| \]

and

(2) The angle bisector of the angle between the vector \( U \) and the vector \( PU \)
is \( X_1 \), a characteristic vector of \( Q \) and \( P \). Thus Matrix \( P \) reflects vector
\( U \) across vector \( X_1 \).
Theorem 2. The vectors $U_n$ and $V_n$ are equally inclined to the vector $X_1$ whose slope is $(\sqrt{5} - 1)/2$.

Corollary. The vectors $V_n$ are mapped into vectors $\sqrt{5} U_n$ by $P$ and the vectors $U_n$ are mapped into $V_n$ by $P$.

7. SOME INTERESTING ANGLES

An interesting theorem is

Theorem 3.

$$\tan^{-1} \frac{L_n}{L_{n+1}} - \tan^{-1} \frac{L_{n+1}}{L_{n+2}} = \frac{(-1)^n}{F_{2n+2}}$$

$$\tan^{-1} \frac{F_n}{F_{n+1}} - \tan^{-1} \frac{F_{n+1}}{F_{n+2}} = \frac{(-1)^{n+1}}{F_{2n+2}}$$

Theorem 4.

$$\tan^{-1} \frac{F_n}{F_{n+1}} = \sum_{m=1}^{n} (-1)^{m+1} \tan^{-1} \frac{1}{F_{2m}}$$

We proceed by mathematical induction. For $n = 1$, it is easy to verify $\tan^{-1} \frac{1}{F_2} = \tan^{-1} (1/F_1)$.

Assume true for $n = k$, that is

$$\tan^{-1} \frac{F_k}{F_{k+1}} = \sum_{m=1}^{k} (-1)^{m+1} \tan^{-1} \frac{1}{F_{2m}}$$

But, by Theorem 3,

$$\tan^{-1} \frac{F_{k+1}}{F_{k+2}} = \tan^{-1} \frac{F_k}{F_{k+1}} + \tan^{-1} \frac{(-1)^k}{F_{2k+2}}$$

Thus, if

$$\tan^{-1} \frac{F_k}{F_{k+1}} = \sum_{m=1}^{k} (-1)^{m+1} \tan^{-1} \frac{1}{F_{2m}}$$

then

$$\tan^{-1} \frac{F_{k+1}}{F_{k+2}} = \sum_{m=1}^{k} (-1)^{m+1} \tan^{-1} \frac{1}{F_{2m}} + \tan^{-1} \frac{(-1)^k}{F_{2k+2}}$$

$$= \sum_{m=1}^{k+1} (-1)^{m+1} \tan^{-1} \frac{1}{F_{2m}}$$

because $\tan^{-1}(-X) = -\tan^{-1} X$ and $(-1)^k = (-1)^{k+2}$ and the proof is complete.
8. AN EXTENDED RESULT

Theorem 5. The series
\[ A = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{\tan^{-1} \frac{1}{F_{2m}}}{F_{2m}} \]
converges and \( A = \tan^{-1} (\sqrt{5} - 1)/2 \).

Proof: Since the series is an alternating series, and, since \( \tan^{-1} x \) is a continuous increasing function, then
\[
\tan^{-1} \frac{1}{F_{2n}} > \tan^{-1} \frac{1}{F_{2n+2}} \quad \text{and} \quad \tan^{-1} 0 = 0.
\]
The angle \( A \) must lie between the partial sums \( S_N \) and \( S_{N+1} \) for every \( N > 2 \) by the error bound in the alternating series, but \( S_N = \tan^{-1} (F_N/F_{N+1}) \). Thus the angles of \( U_N \) and \( U_{N+1} \) lie on opposite sides of \( A \). By the continuity of \( \tan^{-1} x \) then
\[
\lim_{n \to \infty} \tan^{-1} \frac{F_N}{F_{N+1}} = A = \tan^{-1} (\sqrt{5} - 1)/2.
\]

Comment: The same result can be obtained simply from
\[
\tan \left( \tan^{-1} \frac{F_n}{F_{n+1}} - \frac{\sqrt{5} - 1}{2} \right) = (-1)^{n+1} \left( \frac{\sqrt{5} - 1}{2} \right)^{2n+1}.
\]
Which slope gives a better numerical approximation to \( \frac{\sqrt{5} - 1}{2} \), \( F_n/F_{n+1} \) or \( L_n/L_{n+1} \)? Hmm?

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