FURTHER APPEARANCE OF THE FIBONACCI SEQUENCE

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Besides the widespread use of Fibonacci's sequence in Mathematics generally, and the occurrence of the sequence in such diverse fields as electrical network theory and biology (e.g., in the botanical phenomenon of phyllotaxis and the genealogical tree of the male bee [1]), there are certain non-scientific contexts in which its appearance may be of interest.

Both instances to which I shall refer in a moment involve not only the Fibonacci sequences but the ratio known as the Golden Section which has exercised a powerful influence on men's minds down the ages, and about which there is considerable literature. The idea of the Golden Section, probably of Pythagorean origin, is stated by Euclid (Book 2, proposition 11, according to the standard Heath translation) in the following problem: "To cut a given straight line so that the rectangle contained by the whole and one of the segments is equal to the square on the remaining segment." A little calculation reveals that for a segment of unit length, the division (Golden Section) occurs at the irrational point distant $X = (\sqrt{5} - 1)/2 = .62$ from the origin, i.e., X is a solution of the equation $x^2 + x - 1 = 0$.

Now

$$X = \lim_{n\to\infty}\left(\frac{F_n}{F_{n+1}}\right)$$

$$= \lim_{n\to\infty}\left(\frac{H_n}{H_{n+1}}\right)$$

where F_n is the nth term of the ordinary Fibonacci sequence and H_n is the nth term of the generalized Fibonacci sequence [2]. Hence the link between the Golden Section and the Fibonacci sequence.

Psychologists have found by experiment that aesthetically the most pleasing rectangle is the one whose sides are in the ratio $X: 1 - X = (\sqrt{5} - 1): (3 - \sqrt{5}) = 1:X$. Recognizing this aspect of beauty, the ancient Greeks sometimes constructed temples according to these proportions.

A more subtle appreciation of the aethetic qualities of the Golden Section is detailed by Hambidge [3] in his study of Greek vases. After searching inquiries concerning the bases of design in nature and in art, he concludes that the "principle of dynamic symmetry" manifest in shell growth and in leaf distribution in plants was known only to the Egyptians and the Greeks. By meticulous measurements of objects of ancient art, such as Egyptian bas-reliefs and Greek pottery, Hambidge exhibits the constant but hidden occurrence of the Golden Section.

No less meticulous has been the very recent detailed research of Professor G. E. Duckworth [4], of Princeton, into the structural patterns and proportions used by Vergil in the Aeneid. In carefully analyzing the literary architecture of this epic, Duckworth discovers, quite by accident, the basic mathematical symmetry which Vergil consciously used in composing the Aeneid. This is a reminder that ancient poetry was intended to be heard and that, like music, as Duckworth points out, harmony and mathematical proportion appeal to the ear and the imagination.

In his investigations, he gives evidence that

(i) Other poets of Vergil's era, e.g., Catullus, Lucretius, Horace and Lucan, used the Fibonacci sequence in the structure of their poems; and

(ii) Besides the frequent occurrence in the Aeneid of the Golden Section for the ordinary Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ⋯, Vergil also frequently used several Fibonacci sequences, namely, those which in my notation [2] would be labelled H_{12} (the Lucas sequence) H_{13}, H_{14}, H_{15}, H_{23}, H_{34}, H_{45}, H_{67}.

This latter discovery raises a very important point. We are told that Vergil was a serious student of Mathematics. Duckworth produces evidence to show that Vergil, and other poets of his time, were familiar with the Golden Mean and the Fibonacci sequences, a fact which suggests that the Greek and Roman mathematicians knew about the Fibonacci sequence, though there is no record that this is so. [Have we therefore given our Association the correct name?]

Like the work of Hambidge, the minutiae of the painstaking scholarly researches of Duckworth and his fellow-workers reveal a fascinating modern tendency, namely, the successful search for the mathematical expression of beauty and form (and sometimes of chaos). Their discoveries pose a problem

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