ON THE PERIODICITY OF THE LAST DIGITS OF THE FIBONACCI NUMBERS

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In the FIBONACCI QUARTERLY volume 1, number 2, page 84, Stephen P. Geller announced some empirical data on the periodicity of the last digits of the Fibonacci numbers 1, 1, 2, 3, 5, $\cdots$. Using a table of the first 571 Fibonacci numbers, published by S. L. Basin and V. E. Hoggatt, Jr. in RECREATIONAL MATHEMATICS MAGAZINE issue number 11, October 1962, pp. 19–30, he brought out the fact that the last (units) digit of the sequence is periodic with period 60, and that the last two digits are similarly periodic with period 300. Setting up an IBM 1620 he further found that the last three digits repeat every 1,500 times, the last four every 15,000, the last five every 150,000, and finally after the computer ran for nearly three hours a repetition of the last six digits appeared at the 1,500,000th Fibonacci number. Mr. Geller comments: "There does not yet seem to be any way of guessing the next period, but perhaps a new program for the machine which will permit initialization at any point in the sequence for a test will cut down computer time enough so that more data can be gathered for conjecturing some rule for these repetition periods."

I would like to purse half the money necessary to run a computer that will supply the next periods I know. However, since I know the exact period of any number of last digits, the money of the whole world will not suffice. The next period is 15,000,000. Generally the following theorem holds:

Theorem 1. The last $d \geq 3$ digits of the Fibonacci numbers repeat every $15 \cdot 10^{d-1}$ times.

The proof is based on the following theorems from the theory of Fibonacci numbers.

Notation. $A(n)$ – the period of the Fibonacci sequence relative to n.

$a(n)$ – the least positive subscript of the Fibonacci numbers divisible by n (known as "rank of apparition" of n).

$\{a, b, \cdots\}$ – the least common multiple of a, b, $\cdots$.

Theorem 2. $A(n)$ exists for each whole positive n.

Theorem 3. If $n = p_1^{d_1} p_2^{d_2} \cdots p_k^{d_k}$ is the canonical decomposition of n into different prime-powers ($p_1, p_2, \cdots, p_k$ being different primes and $d_1, d_2, \cdots, d_k$ being positive integers), then

$$A(n) = \left\{ A\left(p_1^{d_1}\right), A\left(p_2^{d_2}\right), \cdots, A\left(p_k^{d_k}\right) \right\} .$$

<u>Theorem 4.</u> For any odd prime p and whole positive d,

$$A(p^d) = a(p^d), \quad 2a(p^d), \quad \text{or} \quad 4a(p^d)$$

according as

$$a(p^d) \equiv 2, \quad 0, \quad \text{or} \quad \pm 1 \pmod 4 .$$

For $d \geq 3$,

$$A(2^d) = 2a(2^d) .$$

<u>Theorem 5.</u> For $d \geq 3$, $a(2^d) = 3 \cdot 2^{d-2}$.

For any whole positive d, $a(5^d) = 5^d$.

Proof of Theorem 1. Obviously Geller's problem is equivalent with the one of determining the period of the Fibonacci sequence relative to 10^d for any whole positive $d \geq 3$. Now, by the above theorems,

$$\begin{aligned} A(10^d) = A(2^d 5^d) &= \{A(2^d), A(5^d)\} \\ &= \{2a(2^d), 4a(5^d)\} \\ &= \{2 \cdot 3 \cdot 2^{d-2}, 4 \cdot 5^d\} \\ &= 4\{3 \cdot 2^{d-3}, 5^d\} \\ &= 4 \cdot 3 \cdot 2^{d-3} \cdot 5^d \\ &= 15 \cdot 10^{d-1} . \end{aligned}$$