SOME REMARKS ON CARLITZ' FIBONACCI ARRAY
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Recently in this journal [Vol. 1, No. 2, pp. 17—27] Carlitz defined a Fibonacci array. Among the properties not included in his discussion are the following summation formulas: (Recall $u_{0,n} = F_n$; $u_{1,n} = F_{n+2}$; $u_{r,n} = u_{r-1,n} + u_{r-2,n}$)

(I) \[ \sum_{n=0}^{r} u_{r-n,n} = \frac{2}{5} (r+1)F_{r+1} - F_{r+1}, \]

(II) \[ \sum_{n=0}^{r} (-1)^n u_{r-n,n} = 0, \]

(III) \[ \sum_{n=0}^{r} \binom{r}{n} u_{r-n,n} = \frac{1}{5} [2^{r+1}F_{r+1} - 2], \]

(IV) \[ \sum_{n=0}^{r} (-1)^{n+1} \binom{r}{n} u_{r-n,n} = \begin{cases} 0 & \text{if } r \text{ odd or } r = 0 \\ 2 \cdot 5^{(r-2)/2} & \text{if } r/2 \in \mathbb{Z}^+, \end{cases} \]

The similarities between the formulas above and the four below should be noted:

\[ \sum_{n=0}^{r} F_n F_{r-n} = \frac{1}{5} [rF_r - F_{r+1}], \]

\[ \sum_{n=0}^{r} (-1)^{n+1} F_n F_{r-n} = \begin{cases} 0 & \text{if } r \text{ odd} \\ F_r & \text{if } r \text{ even}, \end{cases} \]

\[ \sum_{n=0}^{r} \binom{r}{n} F_n F_{r-n} = \frac{1}{5} [2^r L_r - 2], \]

\[ \sum_{n=0}^{r} (-1)^{n+1} \binom{r}{n} F_n F_{r-n} = \begin{cases} 0 & \text{if } r \text{ odd or } r = 0 \\ 2 \cdot 5^{(r-2)/2} & \text{if } r/2 \in \mathbb{Z}^+, \end{cases} \]

Because of an overabundance of properties in Carlitz' discussion, we may generalize his array in two ways, taking $H_1 = p$, $H_2 = p + q$. 
We make no attempt to generalize all his results, but consider only the simpler ones. Arabic numerals referring to formulas correspond to those in Carlitz' article.

I. FIRST GENERALIZATION

We define

\( G_{0,n} = H_n \),

\( G_{1,n} = H_{n+2} \),

as the first two rows of the generalized array \( G \). For \( r > 1 \) we define \( G_{r,n} \) by means of

\( G_{r,n} = G_{r-1,n} + G_{r-2,n} \).

It follows that

\( G_{r,n} = pu_{r,n} + qu_{r,n-1} \)

and

\( G_{r,n} = G_{r,n-1} + G_{r,n-2} \).

From these properties Table I is easily computed.

<table>
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<th>( r )</th>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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<td>p</td>
<td>p + q</td>
<td>2p + q</td>
<td>3p + 2q</td>
</tr>
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<td>2</td>
<td></td>
<td>p + 2q</td>
<td>3p + q</td>
<td>4p + 3q</td>
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<td>11p + 7q</td>
</tr>
<tr>
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<td></td>
<td>2p + 3q</td>
<td>5p + 2q</td>
<td>7p + 5q</td>
<td>12p + 7q</td>
<td>19p + 12q</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>3p + 5q</td>
<td>8p + 3q</td>
<td>11p + 8q</td>
<td>19p + 11q</td>
<td>30p + 19q</td>
</tr>
</tbody>
</table>
The symmetry property (5) obviously fails since $G_{0,1} 
eq G_{1,0}$.

If we put

$$(6') \quad g_r(x) = \sum_{n=0}^{\infty} G_{r,n} x^n$$

we find that

$$(7') \quad g_0(x) = \frac{q + px - qx}{1 - x - x^2}, \quad g_1(x) = \frac{p + q + px}{1 - x - x^2}$$

We also have

$$(8') \quad g_r(x) = g_{r-1}(x) + g_{r-2}(x),$$

so that

$$(9') \quad g_r(x) = \frac{H_r + xH_{r+1} + q(F_r - xF_{r+1})}{1 - x - x^2}$$

Putting

$$g(x,y) = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} G_{r,n} x^r y^n$$

we have

$$g(x,y) = \sum_{r=0}^{\infty} \frac{H_r + yH_{r+1} + q(F_r - yF_{r+1})}{1 - y - y^2} x^r$$

so that

$$(11') \quad g(x,y) = \frac{px + py + q - qx + qxy}{(1 - x - x^2)(1 - y - y^2)}$$

It appears that

$$(13') \quad \begin{cases} G_{r+1,r-1} - G_{r,r} = (-1)^r (p - q) \\ G_{r-1,r+1} - G_{r,r} = (-1)^r \end{cases}$$
Indeed, following Carlitz’ procedure we find that

\[
\begin{align*}
(14')
G_{r+2,r-2} - G_{r,r} &= (-1)^{r+1} (p - 2q) \\
G_{r-2,r+2} - G_{r,r} &= (-1)^{r+1} (p + q)
\end{align*}
\]

\[
(15')
\begin{align*}
G_{r+3,r-3} - G_{r,r} &= (-1)^{r} (4p - 6q) \\
G_{r-3,r+3} - G_{r,r} &= (-1)^{r} (4p + 2q)
\end{align*}
\]

and, in general,

\[
(16')
\begin{align*}
G_{r+s,r-s} - G_{r,r} &= (-1)^{r+s+1} F_{F_s - F_{s+1}} q \\
G_{r-s,r+s} - G_{r,r} &= (-1)^{r+s+1} F_{F_s} H_s
\end{align*}
\]

From (16') we note that

\[
(5')
G_{r,n} = G_{n,r} + (-1)^{n} F_{F-r-n} q .
\]

We also note that

\[
(17')
\sum_{r=0}^{n-1} G_{r,r} = \begin{cases} 
2 \cdot F_{F_n} H_n & \text{if } n \text{ even} \\
2 \cdot F_{F_{n+1}} H_{n-1} - q & \text{if } n \text{ odd}
\end{cases}
\]

Among the elementary properties that do not generalize are (10) and (12); however, the latter failure is the basis for the second generalization. The summation formulas in the introduction generalize as

\[
(I') \sum_{n=0}^{r} G_{r-n,n} = \frac{2}{5} [(r + 1)L_{r+1} - F_{F-r+1}] p + \frac{1}{5} [2(r + 1)L_r + F_{F-r+1}] q ,
\]

\[
(II') \sum_{n=0}^{r} (-1)^n G_{r-n,n} = q F_r ,
\]

\[
(III') \sum_{n=0}^{r} \binom{r}{n} G_{r-n,n} = \frac{1}{5} [2^{r+1} L_{r+1} - 2] p + \frac{1}{5} [2^{r+1} L_r + 3] q ,
\]
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(IV') \[
\sum_{n=0}^{r} (-1)^n \binom{r}{n} G_{r-n,n} = \begin{cases} 
q & \text{if } r = 0 \\
5^{(r-1)/2}q & \text{if } r \text{ odd} \\
(-2p + q)5^{(r-2)/2} & \text{if } \frac{r}{2} \in \mathbb{J}_+ 
\end{cases} 
\]

II. SECOND GENERALIZATION

We define

(12'') \[
H_{r,n} = H_{r}^n + H_{r+n} 
\]

It immediately follows that

(1'') \[
H_{0,n} = H_n (q + 1) 
\]

(2'') \[
H_{1,n} = pH_n + H_{n+1} 
\]

(3'') \[
H_{r,n} = H_{r-1,n} + H_{r-2,n} 
\]

(4'') \[
H_{r,n} = H_{r,n-1} + H_{r,n-2} 
\]

(5'') \[
H_{r,n} = H_{n,r} 
\]

See Table II for array H. We also note that

\[
H_{r,n} = p^2 F_r F_n + q^2 F_{r-1} F_{n-1} + pq (F_r F_{n-1} + F_{r-1} F_n) \\
+ pF_{r+n} + qF_{r+n-1} 
\]

We put

(6'') \[
h_r(x) = \sum_{n=0}^{\infty} H_{r,n} x^n 
\]

and see that

(7'') \[
h_0(x) = \frac{H_{0,0} + xH_{-1,0}}{1 - x - x^2}, \quad h_1(x) = \frac{H_{0,1} + xH_{-1,1}}{1 - x - x^2} 
\]
Table II
Array H

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<th>n</th>
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</tbody>
</table>

But by \((3'')\) we have

\((8'')\) \( h_r(x) = h_{r-1}(x) + h_{r-2}(x) \)

so that

\((9'')\) \[ h_r(x) = \frac{H_{0,r} + xH_{-1,r}}{1 - x - x^2} \]

Putting

\[ h(x,y) = \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} H_{r,n} x^r y^n \]
we have
\[ h(x, y) = \sum_{r=0}^{\infty} \frac{H_{0, r} + yH_{-1, r}}{1 - y - y^2} \]
(11")
\[ = \frac{q(1 + q) + (p - q)(1 + q)(x + y) + xy(p^2 - p + 2q + 2pq + q^2)}{(1 - x - x^2)(1 - y - y^2)} \]

From (12") we have
\[ H_{r+s, r-s} - H_{r, r} = H_{r+s}H_{r-s} - H_r^2, \]
so that
\[ (13") \]
\[ H_{r+1, r-1} - H_{r, r} = (-1)^r e, \]
\[ (14") \]
\[ H_{r+2, r-2} - H_{r, r} = (-1)^{r+1} e, \]
\[ (15") \]
\[ H_{r+3, r-3} - H_{r, r} = (-1)^re^4, \]
\[ (16") \]
\[ H_{r+s, r-s} - H_{r, r} = (-1)^{r+s+1}e^sF_r^2, \]
where \( e = p^2 - pq - q^2. \)

The summation formulas previously referred to generalize as
\[ (I") \]
\[ \sum_{n=0}^{r} H_{r-n, n} = (r + 1)H_r + qH_r - \frac{6}{5} F_r + \frac{p}{5} \left[ (H_{r+1} + H_{r-1})p + (H_r + H_{r-2})q \right], \]
\[ (II") \]
\[ \sum_{n=0}^{r} (-1)^n H_{r-n, n} = \begin{cases} 0 & \text{if } r \text{ odd} \\ q(F_{r-1} + qF_{r+1} + 2pF_r) + (p - p^2)F_r & \text{if } r \text{ even} \end{cases}, \]
\[ (III") \]
\[ \sum_{n=0}^{r} \binom{r}{n} H_{r-n, n} = 2^r H_r + \frac{1}{5} [2^r p(H_{r+1} + H_{r-1}) + 2^r q(H_r + H_{r-2}) - 2e], \]
\[ (IV") \]
\[ \sum_{n=0}^{r} (-1)^n \binom{r}{n} H_{r-n, n} = \begin{cases} 0 & \text{if } r \text{ odd} \\ q + q^2 & \text{if } r = 0 \\ -2e^5(r-2)/2 & \text{if } r/2 \in J^+ \end{cases}. \]