ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-189 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$\sum_{r,s=0}^{\infty} \frac{(2r+3s)!}{r!\,s!\,(r+2s)!} \frac{(a-by)^r b^s y^{r+2s}}{(1+y)^{2r+3s+1}} = \frac{1}{1-ay-by^2}$$

H-190 Proposed by H. H. Ferns, Victoria, British Columbia.

Prove the following

$$2^r F_n \equiv n \pmod 5$$

$$2^r L_n \equiv 1 \pmod 5 ,$$

where F_n and L_n are the n^{th} Fibonacci and n^{th} Lucas numbers, respectively, and r is the least residue of $n-1 \pmod 4$.

H-191 Proposed by David Zeitlin, Minneapolis, Minnesota.

Prove the following identities:

(a) $$\sum_{k=0}^{2n} \binom{2n}{k}^3 L_{2k} = L_{2n} \sum_{k=0}^{n} \frac{(2n+k)!}{(k!)^3(2n-2k)!} 5^{n-k}$$

(b) $$\sum_{k=0}^{2n+1} \binom{2n+1}{k}^3 L_{2k} = F_{2n+1} \sum_{k=0}^{n} \frac{(2n+1+k)!}{(k!)^3(2n+1-2k)!} 5^{n+1-k}$$

(c) $$\sum_{k=0}^{2n} \binom{2n}{k}^3 F_{2k} = F_{2n} \sum_{k=0}^{n} \frac{(2n+k)!}{(k!)^3(2n-2k)!} 5^{n-k}$$

(d) $$\sum_{k=0}^{2n+1} \binom{2n+1}{k}^3 F_{2k} = L_{2n+1} \sum_{k=0}^{n} \frac{(2n+1+k)!}{(k!)^3(2n+1-2k)!} 5^{n-k} ,$$

where F_n and L_n denote the n^{th} Fibonacci and Lucas numbers, respectively.

SOLUTIONS

KEEPING THE Q's ON CUE

H-176 Proposed by C. C. Yalavigi, Government College, Mercara, India.

In the "Collected Papers of Srinivas Ramanujan," edited by G. H. Hardy, P. V. Sheshu Aiyer, and B. M. Wilson, Cambridge University Press, 1927, on p. 326, Q. 427 reads as follows:

Show that (corrected)

$$\frac{1}{e^{2\pi}-1} + \frac{2}{e^{4\pi}-1} + \frac{3}{e^{6\pi}-1} + \cdots = \frac{1}{24} - \frac{1}{8\pi} .$$

Provide a proof.

Solution by Clyde A. Bridger, Springfield, Illinois.

A typical term on the left-hand side can be written as

$$\frac{m\,e^{-2m\pi}}{1 - e^{-2m\pi}} = \frac{m\,q^{2m}}{1 - q^{2m}} .$$

This suggests a logarithmic derivative of a product. A suitable well-known product is

(1) $$Q_0 = \prod_{m=1}^{\infty} (1 - q^{2m}) .$$

(See Harris Hancock, Theory of Elliptic Functions, p. 396, Dover, 1958) where (loc cit p. 107)

(2) $$q = \exp(-\pi K'/K) ,$$

in which K and K' have the same relation to elliptic functions as 2π has to trigonometric functions. For example, for the sine-amplitude function, we have

$$\mathrm{sn}(u + 4K + 2i\,K') = \mathrm{sn}\, u$$

and for the sine function,

$$\sin(x + 2\pi) = \sin x .$$

Define K itself as the complete elliptic integral of the first kind

(3) $$K = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}}$$

with modulus $\underline{k}$. Let K', L, and L' be complete elliptic integrals of the first kind with moduli k', ℓ, ℓ', respectively.

The problem now is to find something that contains Q_0 and K. On page 400 (Hancock) appears

$$(kk')^{\frac{1}{12}} = 2^{\frac{1}{6}} q^{\frac{1}{24}} Q_1 Q_3 , \qquad Q_1 Q_2 Q_3 = 1 ,$$

and

$$q^{\frac{1}{8}} \frac{Q_0}{Q_2} = \sqrt{\frac{K}{\pi} \sqrt{kk'}} \quad .$$

Then

$$q^{\frac{1}{12}} Q_0 = 2^{\frac{1}{6}} (kk') \sqrt{\frac{K}{\pi}} \tag{4}$$

is the starting equation.

Suppose that the four elliptic integrals are connected by

$$\frac{nK'}{K} = \frac{L'}{L} \quad , \tag{5}$$

with $k^2 + k'^2 = 1$ and $\ell^2 + \ell'^2 = 1$. (Arthur Cayley, An Elementary Treatise on Elliptic Functions, p. 45, Dover, 1961.)

Then

$$q^n = e^{-\frac{\pi L'}{L}} \tag{2'}$$

and

$$q^{\frac{n}{12}} \prod_{m=1}^{\infty} (1 - q^{2nm}) = 2^{\frac{1}{6}} (\ell\ell')^{\frac{1}{6}} \sqrt{\frac{L}{\pi}} \ . \tag{4'}$$

If we divide Eq. (4) by Eq. (4') and let n 1, we should get 1 = 1. Of the conditions to do this, putting

$$\ell = k' \text{ and } \ell' = k \tag{6}$$

gives a suitable form in n only. We find from Eq. (3) that

(7) $$L = K' \quad \text{and} \quad L' = K .$$

Then Eq. (5) becomes

(5') $$K/K' = \sqrt{n} .$$

Equation (2) becomes

(2'') $$q = e^{-\pi/\sqrt{n}}$$

and Eq. (2') becomes

$$q^n = e^{-\pi/\sqrt{n}} .$$

We can now write the quotient of Eq. (4) by Eq. (4') as

(8) $$e^{-\pi/12\sqrt{n}}(1 - e^{-2\pi/\sqrt{n}})(1 - e^{-4\pi/\sqrt{n}})(1 - e^{-6\pi/\sqrt{n}}) \cdots = n^{\frac{1}{4}} e^{-\pi\sqrt{n}/12}(1 - e^{-2\pi\sqrt{n}})(1 - e^{-4\pi\sqrt{n}})(1 - e^{-6\pi\sqrt{n}}) \cdots$$

Both are infinite products. We now differentiate this logarithmically with respect to n to have

(9) $$\frac{\pi}{24 n\sqrt{n}}\left\{1 - 24\left[\frac{e^{-2\pi/\sqrt{n}}}{1 - e^{-2\pi/\sqrt{n}}} + \frac{2e^{-4\pi/\sqrt{n}}}{1 - e^{-4\pi/\sqrt{n}}} + \cdots\right]\right\} = \frac{1}{4n} - \frac{\pi}{24\sqrt{n}}\left\{1 - 24\left[\frac{e^{-2\pi\sqrt{n}}}{1 - e^{-2\pi\sqrt{n}}} + \frac{e^{-4\pi\sqrt{n}}}{1 - e^{-4\pi\sqrt{n}}} + \cdots\right]\right\}.$$

This reduces readily to

$$1 - 24 \sum_{m=1} m/(e^{2m\pi/\sqrt{n}} - 1) +$$

(9')
$$+ n\left[1 - 24 \sum_{m=1} m/(e^{2m\pi\sqrt{n}} - 1)\right] = \frac{6\sqrt{n}}{\pi} .$$

Now let $n \to 1$. We find the correct solution to be

$$\frac{1}{e^{2\pi} - 1} + \frac{2}{e^{4\pi} - 1} + \frac{3}{e^{6\pi} - 1} + \cdots = \frac{1}{24} - \frac{1}{8\pi} .$$

We have followed Ramanujan's development and have filled in a number of gaps because his procedure is quite esoteric.

Also solved by the Proposer, who used the reference cited in the problem to pick it up at (9').

PARTITION

H-177 Proposed by L. Carlitz, Duke University, Durham, North Carolina. (corrected)

Let $R(N)$ denote the number of solutions of

$$N = F_{k_1} + F_{k_2} + \cdots + F_{k_r} \qquad (r = 1, 2, 3, \cdots),$$

where

$$k_1 \geq k_2 \geq \cdots \geq k_r \geq 1 .$$

Show that

(1) $$R(F_{2n}F_{2m}) = R(F_{2n+i}F_{2m}) = (n - m)F_{2m} + F_{2m-1} \qquad (n \geq m),$$

(2) $$R(F_{2n}F_{2m+1}) = (n - m)F_{2m+1} \qquad (n > m),$$

(3) $$R(F_{2n+1}F_{2m+1}) = (n-m)F_{2m+1} \qquad (n > m),$$

(4) $$R(F^2_{2n+1}) = R(F^2_{2n}) = F_{2n-1} \qquad (n \geq 1).$$

Solution by the Proposer. (See reference below.)

The Proposer has proved that if

$$N = F_{2k} + F_{2k+4} + F_{2k+8} + \cdots + F_{2k+4r-4} \qquad (k \geq 1),$$

then

(*) $$R(N) = kF_{2r} - F_{2r-1} .$$

Also the same result holds for

$$N = F_{2k+1} + F_{2k+5} + \cdots + F_{2k+4r-3} \qquad (k \geq 1).$$

1. Since

$$F_{2n}F_{2m} = F_{2n-2m+2} + F_{2n-2m+6} + \cdots + F_{2n+2m-2} \qquad (n \geq m),$$

it follows from (*) that

$$\begin{aligned} R(F_{2n}F_{2m}) &= (n-m+1)F_{2m} - F_{2m-2} \\ &= (n-m)F_{2m} + F_{2m-1} \qquad (n \geq m). \end{aligned}$$

Since

$$F_{2n+1}F_{2m} = F_{2n-2m+3} + F_{2n-2m+7} + \cdots + F_{2n+2m-1} \qquad (n \geq m),$$

it follows that

$$R(F_{2n+1}F_{2m}) = R(F_{2n}F_{2m}) .$$

L. Carlitz, "Fibonacci Representations," Fibonacci Quarterly, Vol. 6, pp. 193-220.

2. It is proved in Theorem 1 of the paper cited above that if

$$N = F_{k_1} + F_{k_2} + \cdots + F_{k_r} ,$$

where

$$k_1 > k_2 > \cdots \geq k_r \geq 2 ,$$

then

$$(**) \qquad R(N) = R(F_{k_1-k_r+1} + \cdots + F_{k_{r-1}-k_r+1}) + \left(\left[\frac{1}{2}k_r\right] - 1\right) R(F_{k_1-k_r+2} + \cdots + F_{k_{r-1}-k_r+2}) ,$$

and in particular if k_r is odd, then

$$(***) \qquad R(N) = R(F_{k_1-1} + \cdots + F_{k_r-1}) .$$

Since

$$F_{2n}F_{2m+1} = (F_{2n+2m-1} + F_{2n+2m-3} + \cdots + F_{2n-2m+3}) + F_{2n-2m} \quad (n > m) ,$$

it follows from (**) and (***) that

$$\begin{aligned} R(F_{2n}F_{2m+1}) &= R(F_{4m} + F_{4m-4} + \cdots + F_4) + (n - m - 1)\, R(F_{4m+1} + \cdots + F_5) \\ &= (n - m) R(F_{4m} + F_{4m-4} + \cdots + F_4) \\ &= (n - m)(2F_{2m} - F_{2m-2}) = (n - m)F_{2m+1} \qquad (n > m). \end{aligned}$$

3. Since

$$F_{2n+1}F_{2m+1} = (F_{2n+2m} + F_{2n+2m-4} + \cdots + F_{2n-2m+4}) + F_{2n-2m+1} \quad (n \geq m),$$

it follows from (***) and (**) that

$$\begin{aligned}R(F_{2n+1}F_{2m+1}) &= R(F_{2n+2m-1} + F_{2n+2m-5} + \cdots + F_{2n-2m+3}) + F_{2n-2m}) \\ &= R(F_{4m} + F_{4m-4} + \cdots + F_4) + (n - m - 1)R(F_{4m+1} + \\ &\qquad + \cdots + F_5) \\ &= (n - m)R(F_{4m} + F_{4m-4} + \cdots + F_4) \\ &= (n - m)F_{2m+1} \qquad (n \geq m)\,.\end{aligned}$$

4. Since

$$F_{2n+1}^2 = (F_{4n} + F_{4n-4} + \cdots + F_4) + F_2$$

we get

$$\begin{aligned}R(F_{2n+1}^2) &= R(F_{4n-1} + F_{4n-5} + \cdots + F_3) \\ &= R(F_{4n-2} + F_{4n-6} + \cdots + F_2) \\ &= F_{2n} - F_{2n-2} = F_{2n-1} \qquad (n \geq 1)\,.\end{aligned}$$

Similarly, since

$$F_{2n}^2 = F_{4n-2} + F_{4n-6} + \cdots + F_2\,,$$

we have

$$R(F_{2n}^2) = F_{2n-1} \qquad (n \geq 1)\,.$$

WHAT'S THE DIFFERENCE?

H-178 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Put

$$a_{m,n} = \binom{m+n}{m}^2$$

Show that $a_{m,n}$ satisfies no recurrence of the type

$$\sum_{j=0}^{r}\sum_{h=0}^{s} c_{j,k}a_{m-j,n-k} = 0 \qquad (m \geq r,\ n \geq s) ,$$

where the $c_{j,k}$ and r, s are all independent of m, n.

Show also that $a_{m,n}$ satisfies no recurrence of the type

$$\sum_{j=0}^{r}\sum_{k=0}^{n} c_{j,k}\, a_{m-j,n-k} = 0 \qquad (m \geq r,\ n \geq 0) ,$$

where the $c_{j,k}$ and r are independent of m, n.

Solution by the Proposer.

1. Assume that

$$\sum_{j=0}^{r}\sum_{k=0}^{s} c_{j,k}\, a_{m-j,n-k} = 0 \qquad (m \geq r,\ n \geq s) , \tag{1}$$

where $c_{j,k}$ and r, s are independent of m, n.

$$F(x,y) = \sum_{m,n=0}^{\infty} a_{m,n}\, x^m\, y^n .$$

Then we have

$$F(x,y) = \left\{(1 - x - y)^2 - 4xy\right\}^{-\frac{1}{2}} . \tag{2}$$

Indeed,

$$\left\{(1 - x - y)^2 - 4xy\right\}^{-\frac{1}{2}} = (1 - x - y)^{-1}\left\{1 - \frac{4xy}{(1 - x - y)^2}\right\}^{-\frac{1}{2}}$$

$$= \sum_{k=0}^{\infty} \genfrac{}{}{0pt}{}{2k}{k} \frac{(xy)^k}{(1 - x - y)^{2k+1}}$$

$$= \sum_{k=0}^{\infty} \genfrac{}{}{0pt}{}{2k}{k} (xy)^k \sum_{n=0}^{\infty} \genfrac{}{}{0pt}{}{2k + n}{n} (x + y)^n$$

$$= \sum_{k=0}^{\infty} \genfrac{}{}{0pt}{}{2k}{k} (xy)^k \sum_{m,n=0}^{\infty} \genfrac{}{}{0pt}{}{2k + m + n}{m + n} \genfrac{}{}{0pt}{}{m + n}{m} x^m y^n$$

$$= \sum_{m,n=0}^{\infty} x^m y^n \sum_{k=0}^{\min(m,n)} \frac{(m + n)!}{k!\,k!\,(m - k)!\,(n - k)!} .$$

The inner sum is equal to

$$\binom{m + n}{m} \sum_{k}^{\infty} \binom{m}{k}\binom{n}{k} = \binom{m + n}{m}^2 ,$$

which proves (2).)

Now

$$\sum_{j=0}^{r}\sum_{k=0}^{s} c_{j,k} x^j y^k F(x,y) = \sum_{j=0}^{r}\sum_{k=0}^{s} c_{j,k} x^j y^k \sum_{m=0}^{r}\sum_{n=0}^{s} a_{m,n} x^m y^n$$

$$= \sum_{m,n=0}^{\infty} b_{m,n} x^m y^n ,$$

where

$$b_{m,n} = \sum_{j,k} c_{j,k}\, a_{m-j,n-k} \qquad .$$

By (1), we have

$$b_{m,n} = 0 \qquad (m \geq r,\ n \geq r)\ ,$$

so that

$$(3) \qquad \sum_{j=0}^{r} \sum_{k=0}^{s} c_{j,k}\, x^j y^k F(x,y)$$

$$= \sum_{m=0}^{r-1} \sum_{n=0}^{\infty} b_{m,n} x^m y^n - \sum_{m=0}^{\infty} \sum_{n=0}^{s-1} b_{m,n} x^m y^n - \sum_{m=0}^{r-1} \sum_{n=0}^{s-1} b_{m,n} x^m y^n\ .$$

For fixed m, $a_{m,n}$ is a polynomial in n, hence $b_{m,n}$ is also a polynomial in n. Similarly, for fixed n, $b_{m,n}$ is a polynomial in m. Consequently, each of the sums

$$\sum_{m=0}^{r-1} \sum_{n=0}^{\infty} b_{m,n}\, x^m y^n, \qquad \sum_{m=0}^{\infty} \sum_{n=0}^{s-1} b_{m,n}\, x^m y^n$$

is a rational function of x, y. Hence, by (3), $F(x,y)$ is a rational function of x, y. This contradicts (2).

2. Assume that

$$(4) \qquad \sum_{j=0}^{r} \sum_{k=0}^{n} c_{j,k}\, a_{m-n,n-k} = 0 \qquad (m \geq r,\ n \geq 0)\ .$$

Then as in 1, we have

[Continued on page 202.]