A GENERALIZED GREATEST INTEGER FUNCTION THEOREM

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Theorem:

$$\left[a^k F_n + \frac{1}{2}\right] = F_{n+k}, \qquad n \ge k, \quad k \ge 1,$$

where

$$a = \frac{1 + \sqrt{5}}{2}$$

and $[x]$ is the greatest integer contained in x.

Proof. For $k = 1$,

$$\left[aF_n + \frac{1}{2}\right] = F_{n+1} .$$

See [1, Thm. III]. The Binet form for the Fibonacci numbers is

$$F_n = \frac{a^n - b^n}{\sqrt{5}} ,$$

where

$$a = \frac{1 + \sqrt{5}}{2} \text{ and } b = \frac{1 - \sqrt{5}}{2} .$$

Thus

$$a^k F_n = \frac{a^{n+k} - b^n a^k}{\sqrt{5}} = \frac{a^{n+k} - b^n a^k - b^{n+k} + b^{n+k}}{\sqrt{5}}$$

$$= \frac{a^{n+k} - b^{n+k}}{\sqrt{5}} + \frac{b^{n+k} - b^n a^k}{\sqrt{5}}$$

$$= F_{n+k} - b^n\left(\frac{a^k - b^k}{\sqrt{5}}\right) = F_{n+k} - b^n F_k .$$

See [2]. Therefore,

$$a^kF_n + \frac{1}{2} = F_{n+k} + \left(\frac{1}{2} - b^nF_k\right) .$$

The next step is to prove that $\left|b^nF_k\right| < \frac{1}{2}$, $n \geq k$, $k \geq 2$. Since $n \geq k$, let $n = k$ for a fixed k. When $n = k$, $\left|b^nF_k\right|$ will have its largest value. As $n \to \infty$, $\left|b^n\right| \to 0$ monotonically. When k is even:

$$\left|b^kF_k\right| = \left|\frac{b^k(a^k - b^k)}{\sqrt{5}}\right| = \left|\frac{(ba)^k - b^{2k}}{\sqrt{5}}\right| = \left|\frac{1 - b^{2k}}{\sqrt{5}}\right| ,$$

since $ab = -1$. The sequence

$$\left|\frac{1 - b^{2k}}{\sqrt{5}}\right|$$

is monotone increasing, and also

$$\lim_{k \to \infty} \left|\frac{1 - b^{2k}}{\sqrt{5}}\right| = \left|\frac{1}{\sqrt{5}}\right| = \frac{1}{\sqrt{5}} < \frac{1}{2} .$$

Thus,

$$0 \leq \left|b^nF_k\right| < \frac{1}{2}$$

for even k. Now for odd k, we have

$$\left|b^kF_k\right| = \left|\frac{b^k(a^k - b^k)}{\sqrt{5}}\right| = \left|\frac{(ab)^k - b^{2k}}{\sqrt{5}}\right| = \left|\frac{-1 - b^{2k}}{\sqrt{5}}\right|$$

since $ab = -1$. Here we are considering $k = 3, 5, 7, \cdots$. When $k = 3$,

$$b^{2k} = b^6 \approx 0.055726 ;$$

and as k increases, b^{2k} gets smaller rapidly and

$$\left| \frac{-1 - b^{2k}}{\sqrt{5}} \right|$$

becomes smaller. Therefore, if

$$\left| \frac{-1 - b^{2k}}{\sqrt{5}} \right| < \frac{1}{2}$$

for $k = 3$, then it is less than 1/2 for any odd k greater than 3. Thus:

$$\left| \frac{-1 - b^{2k}}{\sqrt{5}} \right| = \left| \frac{1 + b^{2k}}{\sqrt{5}} \right| .$$

If

$$\left| \frac{-1 - b^{2k}}{\sqrt{5}} \right| < \frac{1}{2} ,$$

then

$$\left| 1 + b^{2k} \right| < \frac{\sqrt{5}}{2} \qquad \text{or} \qquad \frac{-\sqrt{5} - 2}{2} < b^{2k} < \frac{\sqrt{5} - 2}{2} .$$

Since $\sqrt{5}$ is approximately 2.2361, the upper bound is approximately 0.1181, and since

$$b^{2k} = b^6 = 0.055726 ,$$

then certainly

$$0 < b^{2k} < \frac{\sqrt{5} - 2}{2} .$$

Therefore:

$$\left|b^k F_k\right| < \frac{1}{2}$$

for all odd k, and, moreover,

$$\left|b^n F_k\right| < \frac{1}{2}$$

for all $k \geq 2$ and $n \geq k$. Finally, since we know that

$$\left|b^n F_k\right| < \frac{1}{2} ,$$

we have

$$-\frac{1}{2} < b^n F_k < \frac{1}{2} .$$

Multiplying by -1 and adding 1/2, we have

$$0 < \frac{1}{2} - b^n F_k < 1 .$$

Since

$$\frac{1}{2} - b^n F_k > 0 ,$$

$$\text{(i)} \qquad a^k F_n + \frac{1}{2} = F_{n+k} + \left(\frac{1}{2} - b^n F_k\right)$$

implies that

$$\left(a^k F_n + \frac{1}{2}\right) > F_{n+k} .$$

Also, since

$$\left(\frac{1}{2} - b^n F_k\right) < 1 ,$$

$$\text{(ii)} \quad F_{n+k} + \left(\frac{1}{2} - b^n F_k\right) < F_{n+k} + 1 \quad \text{and} \quad a^k F_n + \frac{1}{2} < F_{n+k} + 1.$$

Therefore, combining (i) and (ii), we obtain

$$F_{n+k} < a^k F_n + \frac{1}{2} < F_{n+k} + 1$$

or

$$\left[a^k F_n + \frac{1}{2}\right] = F_{n+k} \;.$$

REFERENCES

1. V. E. Hoggatt, Jr., Fibonacci and Lucas Numbers, Houghton Mifflin Company, Boston, 1969, pp. 34-35.
2. V. E. Hoggatt, Jr., John W. Phillips, and H. T. Leonard, Jr., "Twenty-Four Master Identities," The Fibonacci Quarterly, Vol. 9, Feb., 1971, pp. 2-5.

REMARK

With the aid of an ingenious programmer, Galen Jarvinen, it seems reasonable that

$$\left[a^k L_n + \frac{1}{2}\right] = L_{n+k} ,$$

and in general that

$$\left[a^k H_n + \frac{1}{2}\right] = H_{n+k} ,$$

with n somewhat greater than k.