FIBONACCI ONCE AGAIN

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Many popular-type math teasers are based on the concept that may be expressed symbolically as:

$$(\underline{X})(\underline{Y}) = \underline{Y}^2 - \underline{X}^2 .$$

Examples are:

$$34\ 68 = 68^2 - 34^2$$

$$216\ 513 = 513^2 - 216^2 .$$

The true algebraical representation, of course, is:

$$10^{\underline{n}}\underline{X} + \underline{Y} = \underline{Y}^2 - \underline{X}^2$$

$\underline{Y}$ having $\underline{n}$ digits including any initial zero. For example, with $n = 6$, we have:

$$2230\ 047276 = 47276^2 - 2230^2 .$$

Working recently on such examples, it seemed interesting to determine the limiting minimal value of the ratio Y:X, that is of Y/X. This proved quite simple, the derivation being as follows:

For very large values of $\underline{n}$ we may take the maximum value of $\underline{Y}$ as being $10^{\underline{n}}$.

Hence we have

$$10^{\underline{n}}\,\underline{X} + 10^{n} = 10^{\underline{2n}} - \underline{X}^2 .$$

Solving for X,

$$\begin{aligned} 2X &= -10^n + \sqrt{10^{2n} + 4\cdot 10^{2n} - 4\cdot 10^n} \\ &= -10^n + \sqrt{5\cdot 10^{2n} - 4\cdot 10^n} \quad . \end{aligned}$$

Again for very large values we may ignore $4\cdot 10^n$ in the expression under the square-root sign, so having, as $n \to \infty$,

$$2X \to -10^n + 10^n\sqrt{5} \quad ,$$

i.e.,

$$X \to \frac{10^n(\sqrt{5} - 1)}{2} \quad .$$

Hence

$$X/Y \to (\sqrt{5} - 1)/2, \qquad Y/X \to (\sqrt{5} + 1)/2 \quad .$$

Fibonacci again!

It may be noted that with $n = 6$, the greatest value of Y (giving the minimal $X{:}Y$ ratio) gives

$$569466\ 945388 = 945388^2 - 569466^2 \ .$$

And for this we have $Y/X = 1.6601 \cdots$.

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$$\sum_{j=0}^{r}\sum_{k=0}^{\infty} c_{j,k}\, x^j y^k F(x,y) = \sum_{m=0}^{r-1} x^m \sum_{j=0}^{m}\sum_{k=0}^{\infty} c_{j,k}\, y^k \sum_{n=0}^{\infty} a_{m-j,n}\, y^n \ .$$

It follows that $F(x,y)$ is rational in x, again contradicting (2).

Remark. We note that $a_{m,n}$ does satisfy recurrences of the type

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