BACK-TO-BACK: SOME INTERESTING RELATIONSHIPS BETWEEN REPRESENTATIONS OF INTEGERS IN VARIOUS BASES

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A back-to-back relationship between integer representations is one in which the representation of an integer in one base is the reverse of its representation in some other base. Finding such integers and bases is elementary, but the concept does not appear to have received any attention in the literature. A double back-to-back relationship goes one step further: the base indices (written in scale 10 notation) are also the reverses of each other. Examples of single and double back-to-back relationships are:

$$73_{10} = 37_{22}$$

$$169_{82} = 961_{28}$$

Table 1 gives all solutions for integers that have 2, 3, or 4 digits in base-10 notation. The reader may feel tempted to find examples with 5 or more digits. Table 2 lists some of the known double back-to-back examples, leaving a wide open field for the computing-minded enthusiast.

For single back-to-backs we concentrated on finding reverses for base-10 cases. Without that restriction there would be an unlimited number of examples, such as:

$$74_{13} = 47_{22}$$

$$35_{26} = 53_{16}$$

If A, B, C, $\cdots$, represent the digits of an integer N, in base b notation, we seek relationships of the form:

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(1) $$N = (A)(B)(C)\cdots(M)_{10} = (M)\cdots(C)(B)(A)_b ,$$

or solutions to the equation

(2) $$A\cdot 10^{d-1} + B\cdot 10^{d-2} + C\cdot 10^{d-3} + \cdots + M = M\cdot b^{d-1} + \cdots + C\cdot b^2 + B\cdot b + A ,$$

where d represents the number of digits in N. For 2-digit cases we have:

$$(A)(B)_{10} = (B)(A)_b$$

or

(3) $$10A + B = bB + A$$

The solution of (3) is obviously a simple matter. Somewhat more tedious, the 3-digit cases entail integral solutions of

(4) $$100A + 10B + C = b^2C + bB + A .$$

Both the 2-digit and 3-digit cases were found by hand. The lists were checked and confirmed as complete with a Hewlett-Packard 9100A programable calculator — this taking barely two minutes. The same calculator discovered all the 4-digit cases in less than 90 minutes.

The problem of solving Eq. (2) may appear formidable, but there are limits which reduce the amount of numerical work. For a 3-digit case the largest base to be considered is 31. This is so because with $b = 32$, we must have a 4-digit case since $32^2 = 1024$. Similarly the maximum bases for 2, 4, 5, and 6 digits would be 82, 21, 17, and 15, respectively.

Finding solutions for double back-to-backs is more complicated since both the representations and the bases must be in reverse relationship. If a, b, c, $\cdots$, represent the digits of the bases written in base-10 notation, we have

Table 1
SINGLE BACK-TO-BACKS

2-Digit

$13_{10} = 31_4$	$51_{10} = 15_{46}$	$82_{10} = 28_{37}$
$21_{10} = 12_{19}$	$53_{10} = 35_{16}$	$83_{10} = 38_{25}$
$23_{10} = 32_7$	$61_{10} = 16_{55}$	$84_{10} = 48_{19}$
$31_{10} = 13_{28}$	$62_{10} = 26_{28}$	$86_{10} = 68_{13}$
$41_{10} = 14_{37}$	$63_{10} = 36_{19}$	$91_{10} = 19_{82}$
$42_{10} = 24_{19}$	$71_{10} = 17_{64}$	$93_{10} = 39_{28}$
$43_{10} = 34_{13}$	$73_{10} = 37_{22}$	
$46_{10} = 64_7$	$81_{10} = 18_{73}$	

3-Digit

$190_{10} = 091_{21}$	$774_{10} = 477_{13}$
$371_{10} = 173_{16}$	$834_{10} = 438_{14}$
$441_{10} = 144_{19}$	$882_{10} = 288_{19}$
$445_{10} = 544_9$	$912_{10} = 219_{21}$
$511_{10} = 115_{22}$	$961_{10} = 169_{28}$
$551_{10} = 155_{21}$	

4-Digit

$0801_{10} = 1080_9$	$3290_{10} = 0923_{19}$
$1090_{10} = 0901_{11}$	$5141_{10} = 1415_{16}$
$1540_{10} = 0451_{19}$	$7721_{10} = 1277_{19}$
$2116_{10} = 6112_7$	$9471_{10} = 1749_{19}$

(5) $$(A)(B)(C)\cdots(M)_{(a)(b)(c)\cdots(m)} = (M)\cdots(C)(B)(A)_{(m)\cdots(c)(b)(a)} \quad .$$

In order to keep computation within reasonable limits, examples were sought with bases of only two or three digits. A 3-digit integer representation with a 2-digit (in scale-10) base would involve the equation

(6) $A[(a)(b)]^2 + B[(a)(b)] + C$

$$= C[(b)(a)]^2 + B[(b)(a)] + A \ .$$

For example, if $A = 1$, $B = 6$, $C = 9$, $a = 8$, $b = 2$, we have:

$$1[82]^2 + 6[82] + 9 = 9[28]^2 + 6[28] + 1 = 7225 \ ;$$

that is,

$$169_{82} = 961_{28} \ .$$

In Table 2 are listed examples of double back-to-backs. All those in the second part of Table **2** were found by us without calculator aid.

Variations on this type of recreation are endless. Some of the simpler ones could provide classroom enrichment material without entailing too much time on computation. This type of number search could also add zest to the current emphases on modular arithmetic in the so-called "new mathematics."

Table 2
SOME DOUBLE BACK-TO-BACKS

051_{91}	=	150_{19}
144_{73}	=	441_{37}
169_{82}	=	961_{28}
508_{43}	=	805_{34}

If terms in parentheses are considered as single "digits" in the given base we may have examples such as:

$$(1)(12)(7)_{31} = (7)(12)(1)_{13}$$

$$(1)(10)(10)_{41} = (10)(10)(1)_{14}$$

$$(6)(10)(15)_{74} = (15)(10)(6)_{47}$$

$$(10)(0)(16)_{43} = (16)(0)(10)_{34}$$

$(12)(20)(30)_{74} = (30)(20)(12)_{47}$

$(17)(10)(33)_{64} = (33)(10)(17)_{46}$

$(18)(30)(45)_{74} = (45)(30)(18)_{47}$

$(19)(25)(37)_{64} = (37)(25)(19)_{46}$

$(21)(40)(41)_{64} = (41)(40)(21)_{46}$

$(6)(149)(17)_{251} = (17)(149)(6)_{152}$

$(19)(44)(52)_{251} = (52)(44)(19)_{152}$

$(38)(88)(104)_{251} = (104)(88)(38)_{152}$

$(47)(13)(91)_{352} = (91)(13)(47)_{253}$

$(94)(26)(182)_{352} = (182)(26)(94)_{253}$

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$$\sum_{j=0}^{m} \sum_{k=0}^{n} c_{j,k}\, a_{m-j,n-k} = 0 \qquad (m + n > 0) .$$

However this is true of arbitrary $a_{m,n}$ with $a_{00} \neq 0$. We may define $c_{j,k}$ by means of

$$\left(\sum_{m,n=0}^{\infty} a_{mn} x^m y^n \right)^{-1} = \sum_{j,k=0}^{\infty} c_{j,k} x^j y^k .$$

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Commentary on H-169. The theorem is false. Let $a = F_{2n+2}$, $b = c = F_{2n+1}$, $d = F_{2n}$. Thus from $F_{m+1}F_{m-1} - F_m^2 = (-1)^m$, we have $ad - bc = -1$, while $ab + cd = (F_{2n+2}F_{2n+1} + F_{2n}F_{2n+1}) = F_{2n+1}L_{2n+1} = F_{4n+2}$. However, let $N = F_{2n} \neq F_{4n+2}$, so that $F_{2n}^2 + 1 = F_{2n+1}F_{2n-1}$ and $N^2 + 1$ is composite. CONTRADICTION.

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